Power Series Method

- Problem: Solve the following IVP with the power series method around $x_0 = 0$:
$$y''(x) - 2xy'(x) + 8y(x) = 0, \qquad y(0) = \frac{1}{4},\ y'(0) = 0 \tag{1}$$

- Ansatz:
$$y(x) = \sum_{j=0}^{\infty} a_j x^j, \qquad y'(x) = \sum_{j=1}^{\infty} j a_j x^{j-1}, \qquad y''(x) = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} \tag{2}$$
Insertion in the ODE and index substitution to the greatest exponent:
$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} - 2\sum_{j=1}^{\infty} j a_j x^j + 8\sum_{j=0}^{\infty} a_j x^j = 0 \tag{3}$$
$$\sum_{j=0}^{\infty} \left[(j+2)(j+1)a_{j+2} - 2ja_j + 8a_j\right] x^j = 0 \tag{4}$$
Remark: In general the Summation does not run from 0. Then one gets additional equations for the first coefficients.

- Comparision of the coefficients and reposition to the recursion formula for the furthest term of the series:
$$a_{j+2} = \frac{2j-8}{(j+2)(j+1)} a_j, \qquad j \geq 0 \tag{5}$$

- Apply the initial values to decide which series to calculate:
$$0 = y'(0) = a_1 \quad \Rightarrow \quad a_j = 0 \text{ for } j \text{ odd} \tag{6}$$
$$\frac{1}{4} = y(0) = a_0 \quad \Rightarrow \quad \text{Calculate the series with } j \text{ even.} \tag{7}$$

- Calculation of the series:
  1. If the next term in the series vanishes at an index and therefore all further terms as well, the solution is a polynomial. One can calculate the coefficients of this polynomial explicitly.
  2. Recursion formula of the kind $f_{j+a} = b^a f_j$ leads to a solution like $y(x) = z(bx)$.
  3. Those parts of the formula like $f_{j+a} = f_j \prod_{k=0}^{a}(j+k)$ leads to faculty in the explicit formula and therefore the solution are often exponential, trigonometric or hyperbolic functions.
  4. In any other cases one has to calculate a few parameters, guess a formula and prove it by induction (look for hints in the problem).

  Here the first item holds for $j = 4$, thus the first coefficients are
$$a_0 = {}^1\!/_4, \qquad a_2 = -4a_0 = -1, \qquad a_4 = -{}^1\!/_3 a_2 = {}^1\!/_3. \tag{8}$$
  Thus the solution is
$$y(x) = \frac{1}{4} - x^2 + \frac{1}{3}x^4. \tag{9}$$

- Further exercise: Solve the following IVP:
$$y''(x) + 2xy'(x) - y(x) = (1 + x + x^2)e^x, \qquad y(0) = 0,\ y'(0) = \frac{1}{2} \tag{10}$$
Hint: $a_j = \frac{1}{2}\frac{1}{(j-1)!}, \quad j \geq 1$

Laplace Transformation

- Problem: Solve the following IVP with the Laplace transformation

$$y''(x) - y'(x) - 6y(x) = 3x^2 + x - 1, \qquad y(0) = -1,\ y'(0) = 6 \tag{11}$$

- Ansatz:

$$(\mathcal{L}y)(s) = \int_0^\infty \mathrm{e}^{-sx} y(x)\mathrm{d}x \qquad (\mathcal{L}[y'])(s) = s(\mathcal{L}y)(s) - y(0) \tag{12}$$

- Application of the Laplace transformation to both sides of the ODE:

$$\begin{aligned} (\mathcal{L}[y''])(s) - (\mathcal{L}[y'])(s) - 6(\mathcal{L}y)(s) &= \int_0^\infty \mathrm{e}^{-sx}(3x^2 + x - 1)\mathrm{d}x \\ s^2(\mathcal{L}y)(s) - sy(0) - y'(0) - s(\mathcal{L}y)(s) + y(0) - 6(\mathcal{L}y)(s) &= 3\frac{\Gamma(3)}{s^3} + \frac{\Gamma(2)}{s^2} - \frac{1}{s} \end{aligned} \tag{13}$$

- Apply the initial values and reposition to the Laplace transformed function (PFD if needed):

$$(\mathcal{L}y)(s) = \frac{6 + s + s^2 + 7s^3 - s^4}{s^3(s-3)(s+2)} = -\frac{1}{s^3} - \frac{9/5}{s+2} + \frac{4/5}{s-3} \tag{14}$$

- Apply the inverse Laplace transformation:

$$\begin{aligned} y(x) &= -\frac{1}{\Gamma(3)}\left(\left(\mathcal{L}^{-1}\left[\frac{\Gamma(3)}{s^3}\right]\right)\right)(x) - \frac{9}{5}\left(\left(\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\right)\right)(x) + \frac{4}{5}\left(\left(\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]\right)\right)(x) \\ y(x) &= -\frac{1}{2}x^2 - \frac{9}{5}\mathrm{e}^{-2x} + \frac{4}{5}\mathrm{e}^{3x} \end{aligned} \tag{15}$$

  Useful relations to carry out the inverse Laplace transformation:

  1. $\left(\mathcal{L}^{-1}\left[F\left(\frac{s}{a}\right)\right]\right)(x) = a\left(\left(\mathcal{L}^{-1}F\right)(ax)\right)$
  2. $\left(\mathcal{L}^{-1}\left[F(s+a)\right]\right)(x) = \mathrm{e}^{-ax}\left(\left(\mathcal{L}^{-1}F\right)(ax)\right)$
  3. $\left(\mathcal{L}^{-1}\left[\frac{1}{s^n}\right]\right)(x) = \frac{1}{(n-1)!}x^{n-1}$
  4. $\left(\mathcal{L}^{-1}\left[\frac{1}{s^2+(-)a}\right]\right)(x) = \frac{1}{\sqrt{a}}\sin(\mathrm{h})(\sqrt{a}x)$
  5. $\left(\mathcal{L}^{-1}\left[\frac{s}{s^2+(-)a}\right]\right)(x) = \cos(\mathrm{h})(\sqrt{a}x)$
  6. $\left(\mathcal{L}^{-1}\left[F(s)G(s)\right]\right)(x) = \int_0^x \left(\left(\mathcal{L}^{-1}F\right)(x-t)\left(\left(\mathcal{L}^{-1}G\right)(t)\mathrm{d}t\right.\right.$

- Exercise to 6: Calculate a solution to

$$xy(x) + \int_0^x y(x-t)y(t)\mathrm{d}t = 0, \qquad y(0) = -1. \tag{16}$$

- Variation of the problem: Solve the following IVP for $x \geq 0$:

$$\begin{aligned} y''(x) - y'(x) + 2y(x) - z'(x) &= -\mathrm{e}^x, \\ y''(x) - 2y'(x) + y(x) - z'(x) + z(x) &= 0, \\ y'(0) = y(0) = z(0) &= 0. \end{aligned} \tag{17}$$