Complicated Integrals (Partial Fraction Decomposition)

- Problem: Compute the following antiderivative

$$I = \int \frac{4\tan^4 x + 6\tan^3 x + 2\tan^2 x - 4}{2\sin^2 x \tan x + 4\sin^2 x - \sin 2x - 1}\mathrm{d}x \tag{18}$$

- Reorder the function either to a complicated function which derivation will simplify, times another which can easily be integrated or to a complicated function which is only depended of a function which derivative is multiplicated by the first one:

$$\begin{aligned} I &= \int \frac{2\tan^4 x + 3\tan^3 x + \tan^2 x - 2}{\cos^2 x \tan^3 x + 2\cos^2 x \tan^2 x - \tan x \cos^2 x - \cos^2 x - \cos^2 x \tan^2 x}\mathrm{d}x \\ &= \int \frac{2\tan^4 x + 3\tan^3 x + \tan^2 x - 2}{\tan^3 x + \tan^2 x - \tan x - 1}\frac{1}{\cos^2 x}\mathrm{d}x \end{aligned} \tag{19}$$

- Do paritial integration in the first case or the substitution method in the second one.

$$u = \tan x, \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\cos^2 x} \qquad \Rightarrow \qquad I = \int \frac{2u^4 + 3u^3 + u^2 - 2}{u^3 + u^2 - u - 1}\mathrm{d}u \tag{20}$$

- If the power of the numerator is larger than the denominator, do a division of the polynoms:

$$(2u^4 + 3u^3 + u^2 \quad - 2) : (u^3 + u^2 - u - 1) = 2u + 1 + \frac{2u^2 + 3u - 1}{u^3 + u^2 - u - 1} \tag{21}$$

$$\begin{array}{l} 2u^4 + 2u^3 - 2u^2 - 2u \\ \quad u^3 + 3u^2 + 2u - 2 \\ \quad u^3 + u^2 - u - 1 \\ \quad\; + 2u^2 + 3u - 1 \end{array}$$

- Factorize the denominator: $u^3 + u^2 - u - 1 = (u-1)(u+1)^2$
- Use these zero points to make an ansatz of the partial fraction decomposition and compute the coefficients:

$$\frac{2u^2 + 3u - 1}{u^3 + u^2 - u - 1} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{C}{(u+1)^2} \tag{22}$$

$$\begin{aligned} 2u^2 + 3u - 1 &= A(u^2 + 2u + 1) + B(u^2 - 1) + C(u - 1) \\ A + B = 2, \qquad 2A + C &= 3, \qquad A - B - C = -1 \\ A = B = C &= 1 \end{aligned} \tag{23}$$

- Put the results of the division and the decomposition into the integral and evaluate it:

$$\begin{aligned} I &= \int (2u + 1 + \frac{1}{u-1} + \frac{1}{u+1} + \frac{1}{(u+1)^2})\mathrm{d}u \\ &= u^2 + u + \ln|u-1| + \ln|u+1| - \frac{1}{u+1} + C, \qquad C \in \mathbb{R},\ u \neq 1 \end{aligned} \tag{24}$$

- Resubstitute and simplify

$$\begin{aligned} I &= \tan^2 x + \tan x + \ln|\tan x - 1| + \ln|\tan x + 1| - \frac{1}{\tan x + 1} + C \\ &= (\tan x + 1)\tan x + \ln|\tan^2 x - 1| - \frac{1}{\tan x + 1} + C, \quad C \in \mathbb{R},\ \tan x \neq 1 \end{aligned} \tag{25}$$

- Further exercises:

$$\int \frac{2\mathrm{e}^{3x}+3\mathrm{e}^{x}}{\mathrm{e}^{3x}-\mathrm{e}^{2x}+4\mathrm{e}^{x}-4}\mathrm{d}x \tag{26}$$

$$\int \frac{2\ln^2 x-3\ln x}{x\ln^3 x-4x\ln^2 x+5x\ln x-2x}\mathrm{d}x \tag{27}$$

$$\int \frac{\sin x+\cos x}{2-2\sin x+\cos x}\mathrm{d}x \tag{28}$$

$$\int \frac{\sinh^3 x+2\sinh^2 x+5\sinh x+2}{\sinh^4 x\cosh x+2\sinh^3 x\cosh x+2\sinh^3 x\cosh x+2\sinh x\cosh x+\cosh x}\mathrm{d}x \tag{29}$$

$$\int \frac{2x\ln x}{(1+x^2)^2}\mathrm{d}x \tag{30}$$

$$\int \frac{\mathrm{arccot}x}{x^2-4x+4}\mathrm{d}x \tag{31}$$