# Linear maps (Linear System of Equations)

- Problem: For which values of $\alpha \in \mathbb{R}$ contains the set $B = \{\vec{y} = \mathcal{A}\vec{x} \in \mathbb{R}^3 : \vec{x} \in \mathbb{R}\}$ with $\mathcal{A} = \mathcal{I} + \vec{u}\vec{v}^{\mathrm{T}}$ and $\vec{u} = (1, \alpha, 0)^{\mathrm{T}}, \vec{v} = (\alpha, 1, 2)^{\mathrm{T}}$, the vector $\vec{v}$ ($\mathcal{I}$ is the identity matrix of $\mathbb{R}^{3\times 3}$)?

- Formulate the problem in terms of a linear system of equations:
The vector $\vec{u}$ lies in $B$ if and only if the system $\mathcal{A}\vec{x} = \vec{u}$ is solveable.

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1\cdot\alpha & 1\cdot 1 & 1\cdot 2 \\ \alpha\cdot\alpha & \alpha\cdot 1 & \alpha\cdot 2 \\ 0\cdot\alpha & 0\cdot 1 & 0\cdot 2 \end{pmatrix} = \begin{pmatrix} 1+\alpha & 1 & 2 \\ \alpha^2 & 1+\alpha & 2\alpha \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{A}\vec{x} = \vec{u} \leftrightarrow \begin{array}{ccc|c} 1+\alpha & 1 & 2 & \alpha \\ \alpha^2 & 1+\alpha & 2\alpha & 1 \\ 0 & 0 & 1 & 2 \end{array} \tag{32}$$

- Solve the linear system of equations by the Gauß-Jordan method:

$$\mathcal{A}|\vec{u} \rightarrow \begin{array}{ccc|c} 1+\alpha & \mathbf{1} & 0 & \alpha-4 \\ \alpha^2 & 1+\alpha & 0 & 1-4\alpha \\ 0 & 0 & 1 & 2 \end{array} \rightarrow \begin{array}{ccc|c} 1+\alpha & 1 & 0 & \alpha-4 \\ -2\alpha-1 & 0 & 0 & 5-\alpha-\alpha^2 \\ 0 & 0 & 1 & 2 \end{array} \tag{33}$$

From the values in the second row, it is clear that only for $\alpha \neq \frac{1}{2}$ the system can be solved.

- Alternative: One can solve a linear system by calculating the inverse of the matrix $\mathcal{A}$. The existence of it is equivalent to the condition that the determinant of the matrix does not vanish. Therefore in calculating the determinant one can solve this problem, too:

$$\begin{aligned} \det\mathcal{A} &= \det\begin{pmatrix} 1+\alpha & 1 & 2 \\ \alpha^2 & 1+\alpha & 2\alpha \\ 0 & 0 & \mathbf{1} \end{pmatrix} = 1\cdot\begin{pmatrix} 1+\alpha & 1 \\ \alpha^2 & 1+\alpha \end{pmatrix} \\ &= (1+\alpha)^2 - \alpha^2 = 2\alpha + 1 \end{aligned} \tag{34}$$

This does not vanish for $\alpha \neq -\frac{1}{2}$. Thus for every other values $\vec{u}$ lies in $B$.

- Similar problems:
  - Consider the plane $E_\alpha$ given by the points $P = (1, -2, 2)^{\mathrm{T}}$, $Q_\alpha = (\alpha, 0, \alpha)^{\mathrm{T}}$ and $R_\alpha = (1, \alpha-5, 3)^{\mathrm{T}}$ and the line

  $$g_\beta : \vec{x} = \begin{pmatrix} 2\beta \\ 2\beta+2 \\ \beta+2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{R} \tag{35}$$

  with the free parameters $\alpha$, $\beta \in \mathbb{R}$. For which values $\alpha$, $\beta$ have $E_\alpha$ and $g_\beta$ common points?
  - Determine for which values of $\alpha$, $\beta \in \mathbb{R}$ the vector $\vec{b} = (1, 1, \beta)^{\mathrm{T}}$ lies in the range of the linear map $\Phi$ defined by

  $$\Phi(\vec{e}_1) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \qquad \Phi(\vec{e}_2) = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \qquad \Phi(\vec{e}_3) = \begin{pmatrix} 1 \\ \alpha \\ 3 \end{pmatrix}. \tag{36}$$

  Calculate the inverse of $\Phi$ if it exists.

# Systems with Constant Coefficients (Eigenvalues and -vectors)

- Problem: Determine a real-valued fundamental system of the following ODE by transforming it to an equivalent system of first order:

$$x'''(t) - x''(t) + 4x'(t) - 4x(t) = 0 \tag{37}$$

- Set up the relevant matrix: Substitution of $y(t) = x'(t),\ z(t) = y'(t)$ and introduction of the vector $\vec{u}(t) = (x(t), y(t), z(t))^{\mathrm{T}}$ leads to the system $\vec{u}'(t) = \mathcal{A}\vec{u}(t)$ with

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{pmatrix}. \tag{38}$$

- Calculate the eigenvalues of this matrix:

$$\begin{aligned} 0 &= \det(\mathcal{A} - \lambda\mathcal{I}) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -4 & 1-\lambda \end{vmatrix} \\ &= \lambda^2(1-\lambda) + 4 - 4\lambda = (1-\lambda)(\lambda^2+4) \\ &\qquad \Rightarrow \lambda_1 = 1, \lambda_{2,3} = \pm 2i \end{aligned} \tag{39}$$

- Obtain the corresponding eigenvectors: Solving the linear systems of equation for $(\mathcal{A} - \lambda_j\mathcal{I})\vec{u} = 0$,

$$\lambda_1 : \left.\begin{array}{ccc} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 4 & -4 & 0 \end{array}\right| \begin{array}{c} 0 \\ 0 \\ 0 \end{array}, \qquad \lambda_{2,3} : \left.\begin{array}{ccc} \mp 2i & 1 & 0 \\ 0 & \mp 2i & 1 \\ 4 & -4 & 1 \mp 2i \end{array}\right| \begin{array}{c} 0 \\ 0 \\ 0 \end{array}, \tag{40}$$

leads to the eigenvectors:

$$\vec{u}_1 = (1,1,1)^{\mathrm{T}}, \qquad \vec{u}_2 = (-1,-2i,4)^{\mathrm{T}}, \qquad \vec{u}_3 = \vec{u}_2{}^* = (-1,2i,4)^{\mathrm{T}}. \tag{41}$$

- Conclude the general solution: The general complex-valued solution is the linear combination of these eigenvectors times the exponential function with the corresponing eigenvalue as parameter:

$$\vec{u}(t) = c_1\vec{u}_1\mathrm{e}^{t} + c_2\vec{u}_2\mathrm{e}^{2it} + c_3\vec{u}_3\mathrm{e}^{-2it}, \qquad c_1,\ c_2,\ c_3 \in \mathbb{C} \tag{42}$$

In decomposing the exponential functions into sinus and cosinus functions one gets the real-valued solution:

$$\vec{u}(t) = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mathrm{e}^{t} + C_2 \begin{pmatrix} \cos 2t \\ -2\sin 2t \\ -4\cos 2t \end{pmatrix} + C_3 \begin{pmatrix} \sin 2t \\ 2\cos 2t \\ -4\sin 2t \end{pmatrix}, \quad C_1,\ C_2,\ C_3 \in \mathbb{R} \tag{43}$$

Resubstitution leads to the solution of the original problem:

$$x(t) = C_1\mathrm{e}^{t} + C_2\cos 2t + C_3\sin 2t, \qquad C_1,\ C_2,\ C_3 \in \mathbb{R} \tag{44}$$

- Further exercise: Determine the general solution of the following linear system of differential equations:

$$\begin{aligned} x'(t) &= -x(t) + y(t) + z(t) \\ y'(t) &= x(t) - y(t) + z(t) \\ z'(t) &= x(t) + y(t) + z(t) \end{aligned} \tag{45}$$