First Order ODE

Seperable ODE (Separation of Variables)

- Problem: Calculate for a given function $a : [0,1] \to (-1,\infty)$ the solution $y : [0,1] \to \mathbb{R}$ of the following ODE:

$$(a(x)+1)\,y'(x) + \frac{1}{2}a'(x)\,(y(x)+1) = 0 \tag{46}$$

  Is there a number $\lambda \in \mathbb{R}$ with $a(x) = \lambda x$ for which the ODE has a solution with $y(0) = 0$ and $y(1) = -1/2$?

- General recognition feature: The function including their derivatives and its variable can be seperated by reposition the equation.

- Reposition terms dependend on y on one side of the equation and explicit dependencies of x on the other:

$$\frac{y'(x)}{y(x)+1} = -\frac{1}{2}\frac{a'(x)}{a(x)+1} \tag{47}$$

  Remark: Make sure that you won't forget any solution by dividing zero: For every $a(x)$ the function $y(x) = -1$ is a solution of the ODE, too.

- Integrate the equation and use the substitution rule on the left side:

$$\begin{aligned}
\int \frac{1}{y(x)+1}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x &= -\frac{1}{2}\int \frac{1}{a(x)+1}\frac{\mathrm{d}a}{\mathrm{d}x}\mathrm{d}x \\
\int \frac{1}{y(x)+1}\mathrm{d}y &= -\frac{1}{2}\int \frac{1}{a(x)+1}\mathrm{d}a \\
\ln|y(x)+1| &= -\frac{1}{2}\ln|a(x)+1| + C, \qquad C \in \mathbb{R}
\end{aligned} \tag{48}$$

- Reposition the result to $y(x)$:

$$y(x) = c\exp\left(-\frac{1}{2}\ln|a(x)+1|\right) - 1 = \frac{c}{\sqrt{a(x)+1}} - 1, \qquad c \in \mathbb{R} \tag{49}$$

- Calculate the solution for the given IVP in evaluating the constant of integration: For $a(x) = \lambda x$:

$$\begin{aligned}
0 &= y(0) = c - 1 \\
\frac{1}{2} &= y(1) + 1 = \frac{c}{\sqrt{\lambda+1}} \\
\Rightarrow &\quad c = 1,\ \lambda = 3 \\
\Rightarrow &\quad y(x) = \frac{1}{\sqrt{3x+1}} - 1
\end{aligned} \tag{50}$$

Similarity ODE (Substitution)

- Problem: Solve the IVP:

$$(x^2 + xy(x)(y'(x) - 1) = (y(x))^2, \qquad y(1) = 0 \tag{51}$$

- General recognition feature: The differential equation is invariant under the transformation $y(x) \to \lambda y(x)$, $x \to \lambda x$.

- Apply the substitution $z(x) = \frac{y(x)}{x}$ $(y'(x) = xz'(x) + z(x))$:

$$\begin{aligned} x^2(z(x)+1)(xz'(x) + z(x) - 1) &= x^2(z(x))^2 \\ (z(x)+1)xz'(x) &= 1 \end{aligned} \tag{52}$$

- Solve the resulting ODE by Separation of Variables and apply the initial values after resubstitution:

$$\begin{aligned} \frac{1}{2}(z+1)^2 + c_1 = \int (z+1)\mathrm{d}z &= \int \frac{1}{x}\mathrm{d}x = \ln|x| + c_2 \\ z(x) &= \sigma\sqrt{2\ln|x| + c} - 1, \quad c \in \mathbb{R},\ \sigma \in \{-1, +1\} \quad (53) \\ 0 = y(1) = \sigma\sqrt{c} - 1 &\Rightarrow \sigma = +1,\ c = 1 \\ \Rightarrow y(x) &= x\sqrt{2\ln|x| + 1} - x \quad (54) \end{aligned}$$

Bernoulli ODE (Substitution)

- Problem: Determine the solution of the IVP:

$$(y(x))^2 y'(x) - (y(x))^3 \sin x = x\mathrm{e}^{-3\cos x}, \qquad y(0) = \mathrm{e}^{-1} \tag{55}$$

- General recognition feature: The first order ODE can be repositioned to an ODE where the function exists linear as well as to another power α.

- Substitute $z(x) = (y(x))^{1-\alpha}$: In this case α is -2, thus the substitution $(z(x) = (y(x))^3$, $z'(x) = 3(y(x))^2 y'(x))$ leads to:

$$z'(x) - 3z(x)\sin x = 3x\mathrm{e}^{-3\cos x} \tag{56}$$

- The resulting inhomogenous ODE can be solved by Separation of Variables and varying the constant:

$$\begin{aligned} z_{hom}(x) &= c\mathrm{e}^{-3\cos x}, \qquad c \in \mathbb{R} \\ z_p(x) &= z_{hom}(x)\big|_{c \to c(x)} \quad \Rightarrow \quad c'(x) = 3x \\ \Rightarrow \quad z_p(x) &= \frac{3}{2}x^2\mathrm{e}^{-3\cos x} \quad (57) \end{aligned}$$

- Resubstitute and apply the initial values:

$$y(x) = \sqrt[3]{3/2\,x^2 + c}\,\mathrm{e}^{-\cos x}, \qquad c \in \mathbb{R} \tag{58}$$

$$\mathrm{e}^{-1} = y(0) = \sqrt[3]{c}\,\mathrm{e}^{-1} \quad \Rightarrow \quad y(x) = \sqrt[3]{3/2\,x^2 + 1}\,\mathrm{e}^{-\cos x} \tag{59}$$