Reduction of Order (Substitution)

- Calculate the general solution of the following ODE:

$$xy''(x) - y'(x) = 0 \tag{60}$$

- General recognition feature: Either the function or the variable is missing in the ODE
- Substitute the derivatives by a function of the orignial function/variable: Applying the substitution $z(x) = y'(x)$ reduce the ODE to a seperable one:

$$xz'(x) - z(x) = 0 \tag{61}$$

- Solve the ODE with the reduced order and resubstitute to get another ODE. Apply the initial values, if given, and calculate the solution: Seperation of Variables leads to

$$\begin{aligned} y'(x) &= z(x) = c_1 x, \qquad c_1 \in \mathbb{R} && (62) \\ y(x) &= d_1 x^2 + d_2, \qquad d_1,\, d_2 \in \mathbb{R} && (63) \end{aligned}$$

Non Homogeneous ODE: Variation of Constants

- Problem: The homogeneous part of the ODE

$$(x^2+1)y''(x) - 2xy'(x) + 2y(x) = 6(x^2+1)^2 \tag{64}$$

has the solutions $y_1(x) = x$ and $y_2(x) = x^2 - 1$. Determine a particular solution by varying the constants and state the general solution.

- As an Ansatz for the particular solution, replace in the homogeneous solution the constants by functions:

$$y_p(x) = c_1(x)x + c_2(x)(x^2-1) \tag{65}$$

- While calculating the derivatives of the Ansatz, keep in mind, that the derivatives of the replaced constants should only contribute in the highest relevant derivative:

$$\begin{aligned} y_p'(x) &= \underbrace{c_1'(x)x + c_2'(x)(x^2-1)}_{=0} + c_1(x) + 2xc_2(x) \\ y_p''(x) &= c_1'(x) + 2xc_2'(x) + 2c_2(x) && (66) \end{aligned}$$

- Insert the Ansatz in the ODE. Solve the system of first order ODE consisting of this equation combined with the conditions obtained in the last step:

$$\begin{aligned} &\begin{cases} xc_1'(x) + (x^2-1)c_2'(x) = 0 \\ \mathbf{1}c_1'(x) + 2xc_2'(x) = 6(x^2+1)^2 \end{cases} \\ \rightarrow &\begin{cases} -(x^2+1)c_2'(x) = -6x(x^2+1)^2 \\ c_1'(x) + 2xc_2'(x) = 6(x^2+1)^2 \end{cases} \\ \Rightarrow\ & c_2'(x) = 6x \qquad \Rightarrow \qquad c_2(x) = 3x^2 \\ \Rightarrow\ & c_1'(x) = -6x^2 + 6 \qquad \Rightarrow \qquad c_1(x) = -2x^3 + 6x \\ \Rightarrow\ & y_p(x) = (-2x^3+6x)x + 3x^2(x^2-1) = x^4 + 3x^2 && (67) \end{aligned}$$

- Set up the general solution in adding the the determined particular solution to the homogeneous ones:

$$y(x) = x^4 + 3x^2 + c_1 x + c_2(x^2-1) \qquad c_1,\ c_2 \in \mathbb{R} \tag{68}$$

Further examples of ODEs

- Calculate the general solution of the following ODE:

$$y'(x) = \frac{y^2}{x^2 + xy} \tag{69}$$

Note: The solution can not be expressed explicitly.

- Determine the solution of the IVP

$$y'(x) = y(x) + \frac{1+x}{3} y^{-2}, \qquad y(0) = 2. \tag{70}$$

- Solve the IVP

$$(y(x) + y'(x))y'(x) + 2y''(x) = 0, \qquad y(0) = 0,\ y'(0) = 2. \tag{71}$$

- Calculate the solution of the IVP

$$y'(x) + y(x) - y^3(x) = 0, \qquad y(0) = \frac{1}{2}. \tag{72}$$

- Determine the solution of the following IVP:

$$y'(x) = 2 + 2(y(x))^2 + x^2 + (xy(x))^2, \qquad y(0) = 1 \tag{73}$$

- State the general solution of

$$xy''(x) - y'(x) = x^4. \tag{74}$$

- Evaluate the constant $c \in \mathbb{R}$ so that a polynomial of degree 2 will solve the ODE

$$(1 + x)y''(x) + 2xy'(x) = c + x^2. \tag{75}$$

Determine the general solution of the ODE for this constant.