ODE with Constant Coefficients

- State a real fundamental system of the following ODE:
$$y^{(4)}(x) + 2y'''(x) + 2y''(x) + 2y'(x) + y(x) = 0 \tag{76}$$
- General recognition feature: The coefficients of function and their derivatives are constants.
- Taking the Ansatz $y(x) = \mathrm{e}^{\lambda x}$ into account, the characteristic polynomial can be deduced:
$$\lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda + 1 = (\lambda^2+1)(\lambda+1)^2 = 0 \tag{77}$$
- Zero points λ in this polynomial leads to the (complex valued) solution $\mathrm{e}^{\lambda x}$. If multiple zero points occur the missing solution can be gained by multiplying the first solution by a polynomial:
$$y(x) = (c_0 + c_1 x)\mathrm{e}^{-x} + c_2\mathrm{e}^{ix} + c_3\mathrm{e}^{-ix}, \qquad c_0,\ c_1,\ c_2,\ c_3 \in \mathbb{C} \tag{78}$$
- Split the complex-valued exponential function into real and complexed part and define new constants before these parts, which can then be restricted to real values:
$$y(x) = (d_0 + d_1 x)\mathrm{e}^{-x} + \underbrace{(c_2+c_3)}_{d_2}\cos(x) + \underbrace{i(c_2-c_3)}_{d_3}\sin(x), \qquad d_0,\ d_1,\ d_2,\ d_3 \in \mathbb{R}$$
$$\Rightarrow y_1(x) = \mathrm{e}^{-x}, \qquad y_2(x) = x\mathrm{e}^{-x}, \qquad y_3(x) = \cos(x), \qquad y_4 = \sin(x) \tag{79}$$

Non Homogeneous ODE: Undetermined Coefficients

- Problem: Solve the ODE
$$y^{(4)}(x) + 2y'''(x) + 2y''(x) + 2y'(x) + y(x) = \mathrm{e}^{-x} + x\mathrm{e}^{x}. \tag{80}$$
- General recognition feature: The non homogeneous part is a summation of polynomials of degree m times (complex-valued) exponential functions ($\mathrm{e}^{\mu x},\ \mu \in \mathbb{C}$).
- If the non homogeneous part is a sum of functions, look at each function seperately. For each function apply the Ansatz $y(x) = q(x)x^k\mathrm{e}^{\mu x}$, where $q(x)$ is a polynomial of degree m and k is the multiplicity of μ if the exponential function is a solution of the homogneous ODE and 0 otherwise:
$$\begin{array}{rcl|rcl}
 & & \mathrm{e}^{-x} & & & x\mathrm{e}^{x} \\
y_{p1}(x) & = & a_1 x^2 \mathrm{e}^{-x} & y_{p2} & = & (b_0 + b_1 x)\mathrm{e}^{x} \\
y'_{p1}(x) & = & -a_1(x^2-2x)\mathrm{e}^{-x} & y'_{p2}(x) & = & (b_1 + b_0 + b_1 x)\mathrm{e}^{x} \\
y''_{p1}(x) & = & a_1(x^2-4x+2)\mathrm{e}^{-x} & y''_{p2}(x) & = & (2b_1 + b_0 + b_1 x)\mathrm{e}^{x} \\
y'''_{p1}(x) & = & -a_1(x^2-6x+6)\mathrm{e}^{-x} & y'''_{p2}(x) & = & (3b_1 + b_0 + b_1 x)\mathrm{e}^{x} \\
y^{(4)}_{p1}(x) & = & a_1(x^2-8x+12)\mathrm{e}^{-x} & y^{(4)}_{p2}(x) & = & (4b_1 + b_0 + b_1 x)\mathrm{e}^{x}
\end{array} \tag{81}$$
Remark: In case of sinus- or cosinus-functions one either has to split it into two complex functions or to include the other trigonometric function in the Ansatz as well.
- By the comparision of the coefficients the parameters can be calculated:
$$\begin{array}{rcl|rcl}
a_1 & = & \frac{1}{4} & b_0 & = & -\frac{1}{4} \\
 & & & b_1 & = & \frac{1}{8}
\end{array}$$
$$y(x) = \left(\tfrac{1}{8}x - \tfrac{1}{4}\right)\mathrm{e}^{x} + \tfrac{1}{4}\mathrm{e}^{-x} + (d_0 + d_1 x)\mathrm{e}^{-x} + d_2\cos(x) + d_3\sin(x) \tag{82}$$

Euler ODE (Power Ansatz)

- Find the general solution of the following ODE:

$$xy''(x) + 2y'(x) - \frac{6}{x}y(x) = 0 \tag{83}$$

- General recognition feature: After eleminating the x singularities, the coefficients in front of every derivative are x to the power of the degree of the derivative.
- Application of the power ansatz $y(x) = x^\lambda$ leads to the characteristic polynomial:

$$\begin{aligned} \lambda(\lambda-1)x^{\lambda-1} + 2\lambda x^{\lambda-1} - \frac{6}{x}x^\lambda &= 0 \\ \lambda^2 - \lambda + 2\lambda - 6 &= 0 \\ (\lambda+3)(\lambda-2) &= 0 \end{aligned} \tag{84}$$

- From the zero points, the general solution can be deduced:

$$y(x) = c_1 x^2 + c_2 x^{-3} \tag{85}$$

- Further exercises:

$$xy''(x) + 5y'(x) + \frac{5}{x}y(x) = \frac{1}{x^2} \qquad (H04A8) \tag{86}$$

$$x^3y'''(x) + 3x^2y''(x) - \alpha xy'(x) + 2y(x) = 0 \qquad (H03A7) \tag{87}$$

Reduction of order of d'Alembert

- The ODE

$$(1+x^2)y''(x) - 2xy'(x) + \frac{1}{4}\left(5 + \frac{1}{x^2}\right)y(x) = 0, \qquad x > 0 \tag{88}$$

has the solution $y_0(x) = \sqrt{x}$. Determine all other solutions.

- To reduce the order of the ODE use the ansatz of seperation $y(x) = y_0(x)z(x)$ to it: With the product rule of derivation, this leads to

$$\begin{aligned} y'(x) &= y_0'(x)y(x) + y_0(x)y'(x), \qquad y''(x) = y_0''(x)y(x) + 2y_0'(x)y(x) + y_0(x)y''(x), \\ 0 &= z''(x)(1+x^2)y_0(x) + z'(x)\left(2(1+x^2)y_0'(x) - 2xy_0(x)\right) \\ &+ z(x)\left((1+x^2)y_0''(x) - 2xy_0'(x) + \left(\frac{5}{4} + \frac{1}{4x^2}\right)y_0(x)\right). \end{aligned} \tag{89}$$

- Use the condition that $y_0(x)$ is a solution of the original ODE to get an ODE which is not depended on the function. Reduce the order of the remaining ODE and solve it:

$$p(x) = z'(x) \Rightarrow p'(x)(1+x^2)\sqrt{x} = -p(x)((1+x^2)x^{-1/2} - 2x\sqrt{x}) \tag{90}$$

$$\frac{p'(x)}{p(x)} = -\frac{1-x^2}{x+x^3} = \frac{2x}{1+x^2} - \frac{1}{x}$$

$$z'(x) = p(x) = x + \frac{1}{x} \tag{91}$$

$$z(x) = \frac{1}{2}x^2 + \ln x$$

$$y(x) = c_1\sqrt{x} + c_2\left(\frac{1}{2}x^{\frac{5}{2}} + \sqrt{x}\ln x\right), \qquad c_1,\ c_2 \in \mathbb{R} \tag{92}$$