# Diferentiability in $\mathbb{R}^n$

- Problem: Determine for the given function

$$f(x,y) = (x^2+2y^2)\mathrm{e}^{-(x^2+y^2)} \tag{119}$$

  - the domain and ranges,
  - the intersection of the graph of $f$ with the plane y=0,
  - the level curves of $f$,
  - the partial derivatives in the first and second component $\frac{\partial f}{\partial x}(x,y)$,
  - the gradient $\nabla f(x,y)$,
  - the directional derivative $\frac{\partial f}{\partial \vec{d}}(x,y)$ along the direction $\vec{d} = (y,-x)$,
  - the differential $\mathrm{d}f(x,y)$,
  - the tangent mapping $g$ in the point $\vec{x}_0 = (0,0)^\mathrm{T}$ and
  - the Hessian matrix $H_f$ at the point $\vec{x}_0$.

- The domain on the one side specifies which values one can put in the function: Here the function is defined for every real two component vectors (problematic subfunction would be $\sqrt[n]{x}$ $(x \geq 0)$, $\ln x$ $(x > 0)$, $\frac{1}{x}$ $(x \neq 0)$, $\tan x$ $(x \neq n\pi)$, $\arcsin x$ $(-1 \leq x \leq 1), \ldots$): $(x,y) \in \mathbb{R}^2$
The range on the other side determine which values the function can return when an element of the domain is put into it: In this case the exponential function as well as the squares are not negative for every real value of $x$ and $y$ (other relevant subfunctions: $\sqrt[2n]{x} \geq 0$, $|x| \geq 0$, $x^{2n} \geq 0$, $\frac{1}{x^n} \neq 0$, $\cosh x \geq 1$, $\sin x \in [-1..1], \ldots$). Thus, everywhere, the value of the function is positive or zero: $f(x,y) \geq 0$

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}_0^+ \tag{120}$$

**Remark:** A function $u(x)$ can only be put into another function $v(x)$ (written: $(v \circ u)(x)$), if the range of $u$ is the same as the domain of $v$. Then for the derivatives the product and chain rule as well as the connection to the derivative of the inverse function are still valid, but the products may transform into products of matrices.

- To get the intersection of the graph of a function with a plane, solve the system of equations consisting of the function and the equation of plane:

$$\begin{cases} f &= (x^2+2y^2)\mathrm{e}^{-(x^2+y^2)} \\ y &= 0 \end{cases} \rightarrow \begin{cases} f &= x^2\mathrm{e}^{-x^2} \\ y &= 0 \end{cases} \tag{121}$$

Thus the intersection is described by the formula $x^2\mathrm{e}^{-x^2}$.

- The level curves are the intersections with $f(x,y) = const.$ Thus one has to replace $f$ by a constant (e.g. $c_0$) and reposition the function definition to one of the coordinates: Here the equation can not be algebraic reposition to a coordinate variable. Thus the result has to be given implicitly:

$$c_0 = \left(x^2 + 2\big(y(x)\big)^2\right)\mathrm{e}^{-\left(x^2+\left(y(x)\right)^2\right)} \tag{122}$$

- For the partial derivatives calculate the normal derivatives of the function $f(x,y)$ taking the other component as a constant (e.g. $f(x,y) = f_y(x)$):

$$\begin{aligned}\frac{\partial f}{\partial x} &= \Big[2x + (x^2+2y^2)(-2x)\Big]\, \mathrm{e}^{-(x^2+y^2)} = -2x\left[x^2+2y^2-1\right]\mathrm{e}^{-(x^2+y^2)} \quad (123)\\ \frac{\partial f}{\partial y} &= \Big[4y + (x^2+2y^2)(-2y)\Big]\, \mathrm{e}^{-(x^2+y^2)} = -2y\left[x^2+2y^2-2\right]\mathrm{e}^{-(x^2+y^2)} \quad (124)\end{aligned}$$

  **Remark:** If the varied value appears in the boundaries of an integral, one has to use the trick:

$$\left(\int_{a(x)}^{b(x)} f(t)\mathrm{d}t\right)' = f\big(b(x)\big)b'(x) - f\big(a(x)\big)a'(x) \qquad (125)$$

- The gradient is the parital derivatives arranged as a vector:

$$f'(x,y) = \nabla f = \begin{pmatrix} -2x\left[x^2+2y^2-1\right]\mathrm{e}^{-(x^2+y^2)} \\ -2y\left[x^2+2y^2-2\right]\mathrm{e}^{-(x^2+y^2)} \end{pmatrix} \qquad (126)$$

  **Remark:** If the function consists of several components, then the derivative is the transposed gradients of each component written as rows in a matrix. This matrix is called the Jacobi matrix

- Since the gradient shows the variation along the cartesian coordinats, the derivative along any direction can be obtained by taking the inner product:

$$\begin{aligned}\frac{\partial f}{\partial \vec{d}} &= \vec{d}\cdot\nabla f = -2x\left[x^2+2y^2-1\right]\mathrm{e}^{-(x^2+y^2)}d_x - 2y\left[x^2+2y^2-2\right]\mathrm{e}^{-(x^2+y^2)}d_y\\ &= -2xy\mathrm{e}^{-(x^2+y^2)} \qquad (127)\end{aligned}$$

- The differential describes how the function varies under a variation on both coordinates:

$$\begin{aligned}\mathrm{d}f &= (\mathrm{d}x, \mathrm{d}y)^{\mathrm{T}}\cdot\nabla f\\ &= -2x\left[x^2+2y^2-1\right]\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x + -2y\left[x^2+2y^2-2\right]\mathrm{e}^{-(x^2+y^2)}\mathrm{d}y \qquad (128)\end{aligned}$$

- The tangent mapping is the best linear approximation of the function at a point $\vec{x}_0$ (compare with the Taylorseries):

$$g(x,y) = f(\vec{x}_0) + f'(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0) = 0 + \vec{0}\cdot(\vec{x}-\vec{0}) = 0 \qquad \Rightarrow \text{extremum} \qquad (129)$$

- The Hessian matrix is the conclusion of the second parital derivatives:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(0,0) &= \left[2-10x^2-4y^2+8x^2y^2+4x^4\right]\mathrm{e}^{-(x^2+y^2)}\Big|_{(x,y)=(0,0)} = 2\\ \frac{\partial^2 f}{\partial y^2}(0,0) &= \left[4-2x^2-20y^2+4x^2y^2+8y^4\right]\mathrm{e}^{-(x^2+y^2)}\Big|_{(x,y)=(0,0)} = 4\\ \frac{\partial^2 f}{\partial x\partial y}(0,0) &= \left[-12xy+4x^3y+8xy^3\right]\mathrm{e}^{-(x^2+y^2)}\Big|_{(x,y)=(0,0)} = 0\\ \frac{\partial^2 f}{\partial y\partial x}(0,0) &= \left[-12xy+4x^3y+8xy^3\right]\mathrm{e}^{-(x^2+y^2)}\Big|_{(x,y)=(0,0)} = 0\\ \Rightarrow H_f(0,0) &= \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x} \\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \qquad (130)\end{aligned}$$

  **Remark:** If the function is continuous the theorem of Schwarz holds: $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$

- Further functions to examine:
  - Examine the following function, state the extrema and compute the tangential map at the point $\vec{x}_0 = \left(\frac{1}{2}, \frac{1}{3}\right)$ of it:
$$f(x, y) = x^3 y^2 (1 - x - y) \tag{131}$$
  - Determine the domain, range, level curves and the intersection with the plane $E : \vec{v} = \lambda\binom{1}{1}$ of the following function and calculate the Hessian matrix at the extrema:
$$f(x, y) = \frac{1}{y} - \frac{1}{x} - 4x + y \tag{132}$$
  - State the domain and range of the function
$$f(x_1, x_2, x_3, x_4) = \begin{pmatrix} e^{-1} - x_4 e^{x_1 x_2 + x_3} \\ 2{x_1}^2 - x_2 + x_3 - {x_4}^2 \end{pmatrix}, \tag{133}$$
  - Determine the derivative along an arbitary direction $\vec{d}$ at the point $\vec{x}_0 = (1, 1)$ and the derivative of the inverse function at the point $\vec{x}_1 = (1, 1)$ of the function
$$f(x, y) = \begin{pmatrix} x^3 + xy + 1 \\ x + y + y^3 + 1 \end{pmatrix} \tag{134}$$
  - Calculate the Hessian matrix of the following function:
$$f(x, y) = \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2} & , \quad (x, y) \neq \vec{0} \\ 0 & , \quad (x, y) = \vec{0} \end{cases} \tag{135}$$
  - Determine the tangential map of
$$f(x, y) = \sin\left(x e^y\right). \tag{136}$$