Pre-Semester PHYSICS Classical Mechanics and Elektrodynamics

$\vec{F} = m\ddot{\vec{x}}$

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$\int \vec{E} \mathrm{d}\vec{A} = \frac{Q}{\epsilon_0}$

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Part I

Classical Mechanics

Chapter 1

Mathematical Preface

In this chapter the basic mathematical tools needed throughout this lecture will be introduced.

1.1 Functions, Derivatives and Integrals

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The **derivative** of f is defined via

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \,. \tag{1.1}$$

We list some basic functions and their derivatives:

Function $f(x)$	Derivative $f'(x)$
c	0
x^n	nx^{n-1}
e^x	e^x
$\log(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

Rules for derivates for the composition of functions:

Rule	Function $f(x)$	$f'(x)$
Product	$g(x)h(x)$	$g'(x)h(x) + g(x)h'(x)$
Chain	$g(h(x))$	$g'(h(x)) \cdot h'(x)$
Quotient	$\frac{g(x)}{h(x)}$	$\frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$

For the **anti-derivative** $F(x)$ of a function $f(x)$ holds

$$F'(x) = f(x) \qquad \text{or} \qquad \int f(x)dx = F(x) \,. \tag{1.2}$$

Note: $F(x)$ is not unique defined. If $F(x)$ is a anti-derivative of $f(x)$ then $\tilde{F}(x) = F(x) + c$ with $c \in \mathbb{R}$ is another one.
The **definite integral** is given by

$$\int_a^b f(x)dx = F(b) - F(a)\,. \tag{1.3}$$

We also give the rule for integration by parts and

$$\int f(x)g'(x)dx \quad = \quad f(x)g(x) - \int f'(x)g(x)dx\,. \tag{1.4}$$

1.2 Vectors, dot product and vector product

A vector $\vec{x} \in \mathbb{R}^n$ is given by

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\,. \tag{1.5}$$

Its absolute value reads

$$|\vec{x}| = ||\vec{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + \ldots x_n^2}\,. \tag{1.6}$$

For two vectors $\vec{x}, \vec{y}$ the sum and the multiplication with a scalar $\alpha \in \mathbb{R}$ are given by

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + x_n \end{pmatrix} \qquad \text{and} \quad \alpha\vec{x} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}\,. \tag{1.7}$$

For two vectors $\vec{x}, \vec{y}$ the **dot product** (scalar product) is defined via

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} = \sum_{i=1}^{n} x_i y_i = |\vec{x}||\vec{y}| \cos(\phi) = xy \cos(\phi) \tag{1.8}$$

where ϕ is the included angel of $\vec{x}$ and $\vec{y}$. Often the dot between the vectors is left out $\vec{x} \cdot \vec{y} = \vec{x}\vec{y}$. It holds $\vec{x} \cdot \vec{y} = 0$, if $\vec{x} \perp \vec{y}$.
Especially in the $\mathbb{R}^3$ one can define a **cross product** (vector product) for two vectors $\vec{x}$ and $\vec{y}$ by

$$\vec{z} = \vec{x} \times \vec{y} = -\vec{y} \times \vec{x} \tag{1.9}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} . \tag{1.10}$$

The absolute value of $\vec{z}$ is given by

$$|\vec{z}| = |\vec{x}||\vec{y}| \sin(\phi) \tag{1.11}$$

where ϕ is again the included angle. The vector product has the following properties:

(i) $\vec{z}$ is perpendicular two $\vec{x}$ and $\vec{y}$.

(ii) $|\vec{z}|$ can be interpreted as the unsigned area of the parallelogram having $\vec{x}$ and $\vec{y}$ as sides.

(iii) $\{\vec{x}, \vec{y}, \vec{x} \times \vec{y}\}$ is right-handed.

(iv) $\vec{x} \times \vec{y} = 0$ if $\vec{x}||\vec{y}$.

One useful relation for the vector product is

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b}) \tag{1.12}$$

1.3 Polar coordinates

For some problems (e.g. circular motion) it is useful to introduce polar coordinates:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix} \quad \text{or} \quad r = \sqrt{x^2 + y^2}, \qquad \tan(\phi) = \frac{y}{x} \tag{1.13}$$

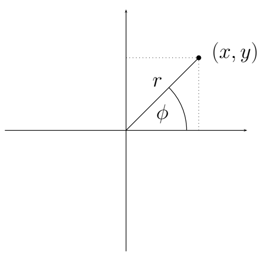


Figure 1.1: Polar coordinates.

Chapter 2

Basic Concepts

2.1 Introduction

The word physics stems form the Greek word $\phi\upsilon\sigma\iota\kappa\eta$, which simple means nature. The aim of physics is to model the inanimate part of our environment in order to make predictions and to understand the laws of nature. This may be visualized as shown in Fig. 2.1. In order to construct such a model a

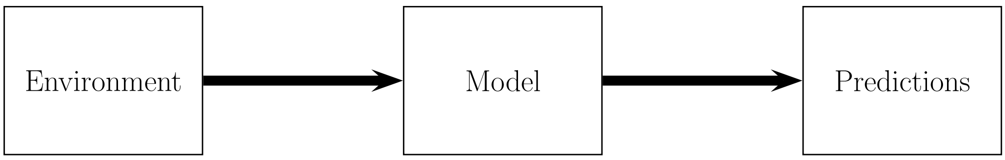


Figure 2.1: Modeling in physics: Events from the inanimate nature are modeled, This models then in turn should serve for further predictions.

physicist avails oneself of mathematics. In setting up a theory one is (in the most cases) interested in having only a few axioms or postulates, from which other things can be derived. For instance–as we will see later on–classical mechanics is fully determined by the three Newton's laws.
Often a theory is valid only in a certain "range", e.g. classical mechanics fails to describe physical processes on the atomic level and has to be replaced or extended to quantum mechanics. Also, if one deals with huge velocities (compared to the light velocity) classical mechanics has to be extended to the special theory of relativity.

2.2 Physical measures

$$\text{Physical Value} = (\text{Absolute Measure}) \cdot (\text{unit}) \tag{2.1}$$

We will use the following shortcuts for the absolute measure:

absolute measure	prefix	shortcut
10^{18}	Exa	E
10^{15}	Peta	P
10^{12}	Tera	T
10^{9}	Giga	G
10^{6}	Mega	M
10^{3}	Kilo	k
10^{-3}	milli	m
10^{-6}	mikro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

2.3 International System of Units

The individual units are arranged via the S.I. system of units (Systeme International des Units):

basic physical value	unit	symbol
Time	second	s
Length	meter	m
Mass	kilogramm	kg

Chapter 3

Kinematics and Kinetics

3.1 Uniform Motion and Linear Motion with Constant Acceleration

The aim of this section will be to describe the motion of a particle, more precisely its position $\vec{x}(t)$ as a function of the time t. Mathematically this is a mapping $[t_1, t_2] \subset \mathbb{R} \to \mathbb{R}^d$ where $d = 1, 2, 3$ is the dimension of the problem.
For simplicity we will start with $d = 1$, that is we are looking for the quantity

$$x(t), \qquad [x(t)] = m\,. \tag{3.1}$$

The **velocity** $v(t)$ of a particle is defined by

$$v(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} \qquad [v(t)] = \frac{m}{s}. \tag{3.2}$$

Derivatives with respect to the time t will in the following be denoted with a dot

$$v(t) = \dot{x}(t)\,. \tag{3.3}$$

In analogy the **acceleration** $a(t)$ of a particle is defined via

$$a(t) = \lim_{h\to 0} \frac{v(t+h) - v(t)}{h} = \dot{v}(t), \qquad [a(t)] = \frac{m}{s^2}\,. \tag{3.4}$$

We are now interested in the case of a constant acceleration, i.e. $a(t) = a$. Integrating equ. (3.4) once we get

$$v(t) = at + v_0 \tag{3.5}$$

with an integration constant v_0. Thus–under a constant acceleration–the velocity increases ($a > 0$) or decreases ($a < 0$) linearly with time. In the case $a = 0$ the velocity stays constant.
To obtain $x(t)$ we integrate (3.5) once more and obtain

$$\boxed{x(t) = \frac{a}{2}t^2 + v_0 t + x_0}. \tag{3.6}$$

This is the path of the particle in one dimension under a constant acceleration. The are two undetermined integration constants v_0 and x_0. Thus to determine the motion of the particle unique one has either to specify the position and the velocity at the same or different times or two positions at different times or two velocities at different times.

EXAMPLE A particle is accelerated with $a = 10\frac{\text{m}}{\text{s}^2}$. At time $t = 0$ the particle is at $x(0) = 0$ and at time $t = 2\text{s}$ at $x(2) = 40\text{m}$. Find $x(t)$!
We have

$$\begin{aligned} 0 &= x(0) = x_0 \\ 40 &= x(2) = 20 + 2v_0 \end{aligned}$$

Thus $x_0 = 0$ and $v_0 = 10$ and we get:

$$x(t) = 5t^2 + 20t$$

These definitions and calculations can easily be generalized to $d = 3$ dimensions. In this case one has (again a constant acceleration $\vec{a}$ is assumed) in vector and component notation

$$\vec{x}(t) = \frac{\vec{a}}{2}t^2 + \vec{v}_0 t + \vec{x}_0 \qquad \text{or} \qquad \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{a_x}{2}t^2 + v_x t + x_0 \\ \frac{a_y}{2}t^2 + v_y t + y_0 \\ \frac{a_z}{2}t^2 + v_z t + z_0 \end{pmatrix} \tag{3.7}$$

EXAMPLE Assume a particle that is at $(0, 0, 0)$ at time $t = 0$ and has only velocity v_x in the x-direction is accelerated through gravity by $-g = -10\frac{m}{s^2}$ in the z-direction. Find $\vec{x}(t)$ and sketch the motion.
For $\vec{x}(t)$ we easily find

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} v_x t \\ 0 \\ -\frac{g}{2}t^2 \end{pmatrix}$$

The last exercise serves as a good example for the so called principle of superposition. We are dealing here with two independent motions: One the one hand we have the motion in x-direction with a constant velocity (i.e. no acceleration) and on the other hand we have have motion in the z-direction with a linearly increasing velocity (i.e. constant accelartion). These two motions are simply added

$$\vec{x}(t) = \begin{pmatrix} v_x t \\ 0 \\ -\frac{g}{2}t^2 \end{pmatrix} = \begin{pmatrix} v_x t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{g}{2}t^2 \end{pmatrix} \tag{3.8}$$

This is called **principle of superposition**.

3.2 Circular motion 1

Another important kind of motion is the **circular motion**. We consider a particle moving on a circle in the $x-y$-plane with radius r. For this kind of

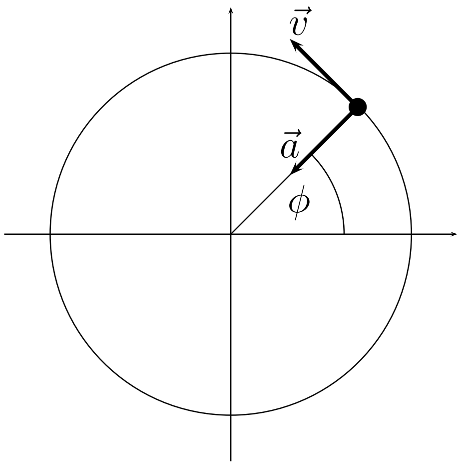


Figure 3.1: Circular motion of a particle.

motion it is more convenient to deal with the **angular velocity** $\omega(t)$:

$$\omega(t) = \dot{\phi}(t), \quad [\omega] = \frac{1}{s} \tag{3.9}$$

We are interested in a constant angular velocity, that is $\ddot{\phi}(t) = \dot{\omega}(t) = 0$ and therefore

$$\omega(t) = \omega = \frac{\Delta\phi}{\Delta t} = \frac{2\pi}{T} \tag{3.10}$$

where T is the **time of circulation**. Often the quantity f defined via

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{T}}, \qquad [f] = \frac{1}{s} \tag{3.11}$$

is used. For $\phi(t)$ we get

$$\phi(t) = \omega\, t + \phi_0 \tag{3.12}$$

If we introduce polar coordinates, set $\phi_0 = 0$ we are able to specify the position of the particle in Cartesian coordinates

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} r\cos(\omega\, t) \\ r\sin(\omega t) \end{pmatrix} . \tag{3.13}$$

For the velocity $\vec{v}(t)$ we obtain

$$\vec{v}(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \omega \begin{pmatrix} -r\sin(\omega\, t) \\ r\cos(\omega t) \end{pmatrix} \tag{3.14}$$

It can be easily seen, that for each time t it holds $\vec{x}(t) \perp \vec{v}(t)$ since

$$\vec{x}(t) \cdot \vec{v}(t) = \begin{pmatrix} r\cos(\omega\, t) \\ r\sin(\omega t) \end{pmatrix} \cdot \begin{pmatrix} -r\sin(\omega\, t) \\ r\cos(\omega t) \end{pmatrix} \tag{3.15}$$

$$= r^2\omega(-\cos(\omega t)\sin(\omega\, t) + \sin(\omega\, t)\cos(\omega t)) = 0 \tag{3.16}$$

If we are interested in the absolute value of the velocity, we have to take the the absolute value of $\vec{v}(t)$ in equ. (3.14). We obtain:

$$\boxed{|\vec{v}(t)| = v = \omega\, r} \tag{3.17}$$

We could have also obtained this result be simply noting the particle needs $T = \frac{2\pi}{\omega}$ going one time around the circle and covering thereby a distance of $s = 2\pi\, r$, such that

$$v = \frac{2\pi\, r}{\frac{2\pi}{\omega}} = \omega\, r\, . \tag{3.18}$$

For the acceleration $\vec{a}$ we finally obtain:

$$\vec{a}(t) = \begin{pmatrix} a_x(t) \\ a_y(t) \end{pmatrix} = -\omega^2 \begin{pmatrix} r\cos(\omega\, t) \\ r\sin(\omega t) \end{pmatrix} = -\omega^2 \vec{x}(t) \tag{3.19}$$

We again see, that $\vec{a}(t) \perp \vec{v}(t)$, and that the acceleration is always pointing in the negative $\vec{x}(t)$ direction. The absolute value $|\vec{a}(t)|$ is given by

$$\boxed{a = \omega^2 r} \tag{3.20}$$

This is called **centripetal acceleration**.

3.3 Newton's axioms

So far we dealt only with the question how to describe the motion of a mass point (that is the path $\vec{x}(t)$ of the mass point). This chapter addresses the question, what makes a body moving and changing its state of motion. The answer is: A **force** exactly does this. Isaac Newton (1643-1727) realized first that for a given body the absolute value of the force is proportional to the absolute value of the acceleration. The constant of proportionality is called the mass m. This is **Newton's second axiom** (the principle of action):

$$\boxed{\vec{F} = m\ddot{\vec{x}}} \qquad [m] = kg,\ [F] = N = kg\frac{m}{s^2} \tag{3.21}$$

If we define **momentum** as $\vec{p} = m\dot{\vec{x}} = m\vec{v}$, we have

$$\vec{F} = \dot{\vec{p}}, \qquad [\vec{p}] = kg\frac{m}{s}\,. \tag{3.22}$$

Even though the last two equations define the mass m, for practical purposes it is, however more important to ask the following two questions:

(a) What is $\ddot{\vec{x}}$, if $\vec{F}$ and m are given?

(b) What is $\vec{F}$, if $\ddot{\vec{x}}$ and m are given?

We will mainly address the first question during this lecture. If the force $\vec{F}$ in equ. (3.21) is zero, we see that $\vec{v}$ is constant. This is **Newton's first axiom** (principle of inertia):
Each body, on which no force acts stays at rest or moves with a constant velocity. In short notation this axiom may be written as

$$\boxed{\sum_i \vec{F}_i = 0 \qquad \Leftrightarrow \qquad \vec{v} = \text{const}}\,. \tag{3.23}$$

In addition **Newton's third axiom** (actio = reactio) states, that if a body 1 acts on a body 2 with a force $\vec{F}_{12}$, then body 2 acts on body 1 with a force $\vec{F}_{21}$, such that

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}\,. \tag{3.24}$$

3.4 Forces

In this section we will introduce the most common forces. At first it should be emphasized that for forces as well as for velocities holds the **principle of**

superposition, i.e. forces may be composed and decomposed.

Gravity
The gravitational force between two bodies with masses m_1, m_2 separated at a distance r is given by:

$$\vec{F} = -\gamma \frac{m_1 m_2}{r^2} \hat{\vec{r}} \qquad \text{or} \qquad \boxed{F = \gamma \frac{m_1 m_2}{r^2}} \tag{3.25}$$

with the unit vector $\hat{\vec{r}} = \frac{\vec{r}}{r}$ and the gravitation constant

$$\gamma = 6,67 \times 10^{-11} \frac{m^3}{kg\, s^2} \tag{3.26}$$

If we are interested in the absolute value of the gravitational force near the earth we set $m_2 = m_e = 5,97 \times 10^{24}\,\text{kg}$ and $r = r_e = 6370\,\text{km}$ and get

$$\boxed{F_G = \frac{\gamma\, m_e}{r_e^2} m \equiv mg} \tag{3.27}$$

with $g = 9,81 \frac{m}{s^2}$ (in the exercises we will often use $g = 10 \frac{m}{s^2}$).

EXAMPLE

A common example is the **inclined plane**. We decompose the gravitational

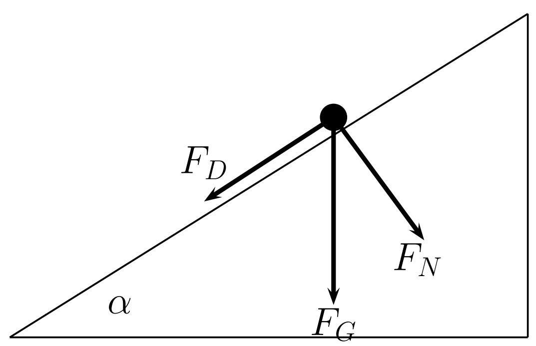


Figure 3.2: Decomposition of forces on an inclined plane.

force in the two components normal force and downhill-slope force. These are given by

$$F_D = F_G \sin\alpha \tag{3.28}$$
$$F_N = F_G \cos\alpha \tag{3.29}$$

Hook's law
The force of a spring is given by (in $d = 1$):

$$\boxed{F = -Dx} \qquad [D] = \frac{N}{m} \tag{3.30}$$

with the spring constant D.

Friction
We will consider two kinds of friction forces.

(i) Static friction: It is given by

$$F_S = \mu\, F_N \tag{3.31}$$

As an example we consider again a body with mass m on an inclined plane. Again, the following forces are relevant:

$$F_D = mg \sin\alpha \tag{3.32}$$
$$F_N = mg \sin\alpha \tag{3.33}$$

The body just starts to move if $F_D = \mu F_N$, that is

$$m\, g \sin\alpha = \mu\, m\, g \cos\alpha \qquad \text{or} \quad \mu = \tan\alpha\,. \tag{3.34}$$

(ii) Stokes friction: It is (phenomenologically) given by

$$F_R = -\gamma_S\, v \tag{3.35}$$

This friction will play a role when we discuss damped oscillations.

Centripetal and Centrifugal Force
In section 3.2 we saw, that an object moving on a circle with some constant velocity v has a changing direction of motion, i.e. the vectot $v(t)$ changes its direction. The rate of this change in velocity is the centripetal acceleration $a_Z = \omega^2 r = \frac{v^2}{r}$ with $v = \omega r$. The corresponding force reads:

$$F_c = \frac{mv^2}{r} \tag{3.36}$$

3.5 Work, Energy, Momentum and their conservation laws

Generally **work** is defined by

$$W = \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}, \quad [W] = J \tag{3.37}$$

We again constrict ourself to the one dimensional case where this formula reduces to

$$\boxed{W = \int_{x_1}^{x_2} F(x)\, dx}. \tag{3.38}$$

We give two (important) examples:

(i) Potential energy of a body with mass m under the influence of gravitation (the body is uplifted from 0 to h.)

$$E_{\text{pot}} = \int_0^h mg\, dx = mgh \tag{3.39}$$

(ii) Potential energy of a body with mass m attached to a spring with spring constant D:

$$E_{\text{pot}} = \int_0^s Dx\, dx = \frac{1}{2} D s^2 \tag{3.40}$$

(iii) Energy that is needed to accelerate a particle with mass m from 0 to v_0:

$$E_{\text{kin}} = \int F ds = \int m \frac{dv}{dt} ds = m \int_0^{v_0} v dv = \frac{m}{2} v_0^2 \tag{3.41}$$

We already entitled the individual works with the common terms.
We will now turn to **energy** and the **conservation of energy**. We consider a particle with mass m subject to a force $F = F(x)$ in one dimension, i.e. its motion is governed by the second Newton's law $F(x(t)) = m\ddot{x}(t)$. Let us define

$$V(x) = -\int F(x)\, dx \qquad \Leftrightarrow \quad -V'(x) = F(x) \tag{3.42}$$

where $V(x)$ is called **potential**. With this definition Newtons second law reads:

$$-V'(x(t)) = m\ddot{x}(t) \tag{3.43}$$

We multiply this equation by $\dot{x}(t)$ and use the chain rule

$$\begin{aligned} -V'(x(t))\dot{x}(t) &= m\ddot{x}(t)\dot{x}(t) \quad (3.44)\\ \Leftrightarrow \quad \frac{d}{dt}\Big(\frac{m}{2}\dot{x}^2 + V(x)\Big) &= 0 \quad (3.45) \end{aligned}$$

Thus, the expression in the brackets is constant (as a function of time) and called **energy** E. Using $v = \dot{x}$, we can write

$$\boxed{E = \frac{m}{2}v^2 + V(x)} \tag{3.46}$$

Again we will re assume above named examples:

(i) Gravitation:

$$F(x) = -mg \quad \Rightarrow \quad V(x) = mgx \quad \Rightarrow \quad E = \frac{m}{2}v^2 + mgx \tag{3.47}$$

(ii) Spring:

$$F(x) = -Dx \quad \Rightarrow \quad V(x) = \frac{D}{2}x^2 \quad \Rightarrow \quad E = \frac{m}{2}v^2 + \frac{D}{2}x^2 \tag{3.48}$$

Next we are going to consider the **momentum** of a system of two interacting particles. Newton's third law tells us, that $\vec{F}_{12} = -\vec{F}_{21}$, that is

$$m_1\ddot{\vec{x}}_1 + m_2\ddot{\vec{x}}_2 = 0 \quad \Rightarrow \quad \frac{d}{dt}\Big(m_1\dot{\vec{x}}_1 + m_2\dot{\vec{x}}_2\Big) = 0 \tag{3.49}$$

Thus we can conclude, that the **total momentum**

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = m_1\dot{\vec{x}}_1 + m_2\dot{\vec{x}}_2 \tag{3.50}$$

is conserved.
Another quantity which we shell define at this point is the so called **center of mass**:

$$\vec{R} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} = \frac{1}{M}(m_1\vec{x}_1 + m_2\vec{x}_2) \tag{3.51}$$

with the total mass $M = m_1 + m_2$. Since

$$\dot{\vec{R}} = \frac{1}{M}(m_1\dot{\vec{x}}_1 + m_2\dot{\vec{x}}_2) = \frac{1}{M}(\vec{p}_1 + \vec{p}_2) = \frac{1}{M}\vec{P} \tag{3.52}$$

and $\vec{P}$ is a constant of motion we can conclude that the center of mass is either at rest or moving with a constant velocity.
This can be easily generalized to a system with N particles (a two body interaction is assumed). Since $\vec{F}_{ik} = -\vec{F}_{ki}$ we have $\sum_i \sum_{k\neq i} \vec{F}_{ik} = 0$. Since the force acting on particle i is $\vec{F}_i = \sum_{k\neq i} \vec{F}_{ki} = \dot{\vec{p}}_i$ we have

$$\dot{\vec{P}} = \sum_i \dot{\vec{p}}_i = \sum_i \sum_{k\neq i} \vec{F}_{ik} = 0 \tag{3.53}$$

and therefore we find a again that the total momentum $\vec{P}$ is a constant of motion. Analogous, the **center of mass**

$$\vec{R} = \frac{\sum_i m_i \vec{x}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{x}_i}{M} \tag{3.54}$$

is a constant of motion.

3.6 Applications: Collisions

In order to discuss an application we will consider the collision of two particles one dimension, i.e. a process

$$v_1, v_2 \rightarrow v_1', v_2' \,. \tag{3.55}$$

There are at least two possible collision processes. For each of them other conservation laws hold.

ELASTIC COLLISION	INELASTIC COLLISION
Conservation of Momentum $m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$	Conservation of Momentum $m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$
Conservation of energy $\frac{m_1}{2}v_1^2 + \frac{m_2}{2}v_2^2 = \frac{m_1}{2}v_1'^2 + \frac{m_1}{2}v_2'^2$	Energy is not conserved: Some energy is converted into heat $\frac{m_1}{2}v_1^2 + \frac{m_2}{2}v_2^2 = \frac{m_1}{2}v_1'^2 + \frac{m_1}{2}v_2'^2 + Q$

Taking into account the conservation of momentum and energy for the **elastic collision** we obtain:

$$m_1(v_1 - v_1') = m_2(v_2' - v_2) \tag{3.56}$$

$$m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2) \tag{3.57}$$

Using the third binomial law we get

$$m_1(v_1 - v_1') = m_2(v_2' - v_2) \tag{3.58}$$
$$m_1(v_1 - v_1')(v_1 + v_1') = m_2(v_2' - v_2)(v_2' + v_2)\,. \tag{3.59}$$

Thus we have

$$v_1 + v_1' = v_2 + v_2' \tag{3.60}$$

and together with the conservation of momentum

$$v_1' = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \quad \text{and} \quad v_2' = \frac{(m_2 - m_1)v_2 + 2m_1v_1}{m_1 + m_2}\,. \tag{3.61}$$

In the case of an **inelastic collision** we will only consider the special case of $v_2 = 0$ and that the two bodies stick together right after the collision, i.e. they are moving with the same velocity $v_1' = v_2'$. Such a collision is called **total inelastic collision**. Thus, the conservation laws

$$m_1v_1 = (m_1 + m_2)v_1' \tag{3.62}$$
$$\frac{m_1}{2}v_1^2 = \frac{m_1 + m_2}{2}v_1'^2 + Q \tag{3.63}$$

yield

$$v_1' = \frac{m_1}{m_1 + m_2}v_1 \quad \text{and} \quad Q = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}v_1^2\,. \tag{3.64}$$

3.7 Circular motion 2

We consider again a particle moving around some axis, which we will call $\vec{\omega}$. For the velocity we obtain:

$$\vec{v}(t) = \vec{\omega} \times \vec{x}(t) \tag{3.65}$$

If the vector $\vec{x}(t)$ is in a plane perpendicular to the axis ω this formula simplifies to $v = \omega r$, where $r = |\vec{x}|$ is the distance from the axis to the particle. Two quantities to be defined in this context are the **angular momentum** $\vec{L}$ and the **moment of torque** $\vec{M}$

$$\vec{L} = \vec{x} \times \vec{p} \qquad \text{and} \qquad \vec{M} = \vec{x} \times \vec{F} \tag{3.66}$$

Since $\dot{\vec{x}}$ is parallel to $\vec{p} = m\dot{\vec{x}}$ and with $\dot{\vec{p}} = \vec{F}$ we obtain

$$\dot{\vec{L}} = \vec{M} \tag{3.67}$$

The kinetic energy of this particle may with help of equ. (3.65) be rewritten as

$$E_r = \frac{m}{2}\vec{v}^2 = \frac{m}{2}(\vec{\omega} \times \vec{x})^2 = \frac{m}{2}\omega^2 r^2 \tag{3.68}$$

where E_r denotes the **rotational energy**. In the last step we assumed again the axis perpendicular to the plane of motion. Defining the **moment of inertia** Θ by:

$$\boxed{\Theta = mr^2} \tag{3.69}$$

we can write

$$\boxed{E_r = \frac{\Theta}{2}\omega^2} \tag{3.70}$$

3.8 Planetary motion

The Kepler problem deals with the motion of two masses m_1 and m_2 under the influence of the gravitational force

$$\vec{F}_{12} = -\gamma \frac{m_1 m_2}{|\vec{x}_1 - \vec{x}_2|^3}(\vec{x}_1 - \vec{x}_2) = -\vec{F}_{21}\,. \tag{3.71}$$

Kepler's laws contain the following statements:

(1) The orbit of a planet about a star is an ellipse with the star at one focus.

(2) A line joining a planet and its star sweeps out equal areas during equal intervals of time. This is also known as the law of equal areas.

(3) The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits.

We will first prove the second statement. Since $\vec{F} \propto \vec{x}$, we see from equ. (3.66) that $\vec{M} = 0$ and consequently $\vec{L}$ is a constant of motion. In particular that tells us, that the motion takes place in a plane. From Fig. 3.4 we can read of that

$$dA = \frac{1}{2}|\vec{x} \times d\vec{x}|\,. \tag{3.72}$$

Taking the time derivative we obtain

$$\dot{A} = \frac{dA}{dt} = \frac{1}{2}|\vec{x} \times \frac{d\vec{x}}{dt}| = \frac{1}{2m}|\vec{x} \times \vec{p}| = \frac{L}{2m}\,. \tag{3.73}$$

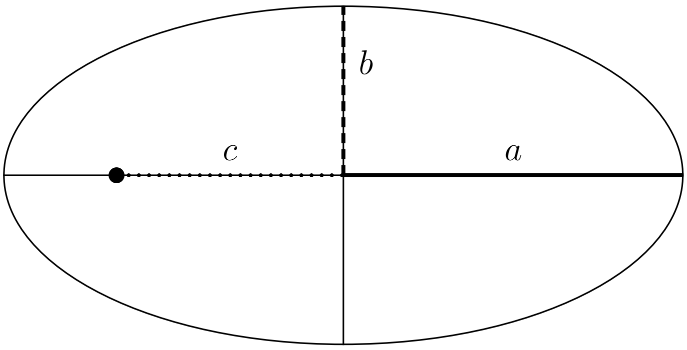


Figure 3.3: Ellipse with major axis a, minor axis b and the linear eccentricity $c = \frac{1}{2}\sqrt{a^2 - b^2}$.

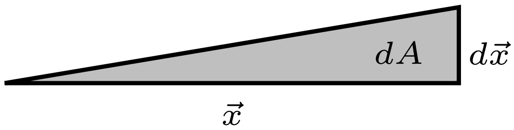


Figure 3.4: Derivation of Newton's second law. The solid filled triangular is the area $dA = \frac{1}{2}|\vec{x} \times d\vec{x}|$.

For the first and third law let us first recall the definition of the center of mass $\vec{R} = \frac{1}{M}(m_1\vec{x}_1 + m_2\vec{x}_2)$. If we furthermore introduce the relative distance $\vec{x} = \vec{x}_1 - \vec{x}_2$ between the two masses we can express the individual positions through this quantities by

$$\vec{x}_1 = \vec{R} + \frac{m_2}{M}\vec{x} \tag{3.74}$$

$$\vec{x}_2 = \vec{R} - \frac{m_1}{M}\vec{x}\,. \tag{3.75}$$

Since the center of mass $\vec{R}$ is a constant of motion we can choose our coordinate system such that the origin coincides with the center of mass. We obtain

$$\vec{x}_1 = +\frac{m_2}{M}\vec{x} \tag{3.76}$$

$$\vec{x}_2 = -\frac{m_1}{M}\vec{x}\,. \tag{3.77}$$

Thus, the equation of motion

$$m_1\ddot{\vec{x}}_1 = -\gamma\frac{m_1 m_2}{|\vec{x}_1 - \vec{x}_2|^3}(\vec{x}_1 - \vec{x}_2) \tag{3.78}$$

may be rewritten as

$$\frac{m_1 m_2}{M}\ddot{\vec{x}} = -\gamma\frac{m_1 m_2}{x^3}\vec{x} \tag{3.79}$$

with $x = |\vec{x}|$. Since the angular momentum $\vec{L}$ is a constant of motion the motion takes in a plane, which we will choose in the $x-y$ plane. Furthermore we will only prove a special (and easier) case of an elliptic motion–the circular motion. We make the ansatz

$$\vec{x}(t) = \begin{pmatrix} a\cos(\omega t) \\ a\sin(\omega t) \\ 0 \end{pmatrix} \quad \Rightarrow \quad \ddot{\vec{x}}(t) = -\omega^2\begin{pmatrix} a\cos(\omega t) \\ a\sin(\omega t) \\ 0 \end{pmatrix} . \tag{3.80}$$

With$|\vec{x}| = a$ this yields

$$-\frac{1}{m_1 + m_2}\omega^2 = -\gamma\frac{1}{a^3} , \tag{3.81}$$

and with $\omega = \frac{2\pi}{T}$ (T being the time of circulation) we finally obtain

$$\frac{T^2}{a^3} = \frac{4\pi^2}{\gamma(m_1 + m_2)} . \tag{3.82}$$

That proves the first and third law.

3.9 Motion of a rigid body

As a first step we define a rigid body as N discrete masses $m_i,\ i = 1, \dots, N$ with pairwise fixed distances. The positions of individual masses will be denoted by $\vec{x}_i$. With that definitions the kinetic energy takes the form

$$E = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{x}}_i^2 . \tag{3.83}$$

Writing $\vec{x}_i = \vec{R} + \vec{x}_i'$ we obtain

$$E = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{R}}^2 + \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\vec{x}}_i')^2 + \dot{\vec{R}}\sum_{i=1}^{N} m_i\dot{\vec{x}}_i' . \tag{3.84}$$

The last sum (without the $\dot{\vec{R}}$ in front) yields with $\vec{x}_i' = \vec{x}_i - \vec{R} = \vec{x}_i - \frac{1}{M}\sum_{i=1}^N m_i\vec{x}_i$

$$\sum_{i=1}^N m_i\dot{\vec{x}}_i - \sum_{i=1}^N m_i\dot{\vec{R}} = \sum_{i=1}^N m_i\dot{\vec{x}}_i - \dot{\vec{R}}M = \sum_{i=1}^N m_i\dot{\vec{x}}_i - \sum_{i=1}^N m_i\dot{\vec{x}}_i = 0 \quad (3.85)$$

and thus

$$E = \frac{1}{2}\left(\sum_{i=1}^N m_i\right)\dot{\vec{R}}^2 + \frac{1}{2}\sum_{i=1}^N m_i(\dot{\vec{x}}_i')^2\,. \quad (3.86)$$

Assuming the body rotating around some axis $\vec{\omega}$ we can write $\dot{\vec{x}}_i' = \vec{v}_i' = \vec{\omega}\times\vec{x}_i'$. Since $\vec{a}^2 = |\vec{a}|^2$ and if we define r_i' as the distance from the axis of rotation to the position $\vec{x}_i'$, we can write $(\vec{\omega}\times\vec{x}_i')^2 = \omega^2(r_i')^2$. The energy takes the common form

$$E = \frac{1}{2}\left(\sum_{i=1}^N m_i\right)\dot{\vec{R}}^2 + \frac{1}{2}\omega^2\sum_{i=1}^N m_i(r_i')^2 \quad (3.87)$$

$$= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\Theta\omega^2\,. \quad (3.88)$$

where we defined the **moment of inertia**

$$\Theta = \sum_{i=1}^N m_i r_i^2\,. \quad (3.89)$$

Since the movement of the rigid body can be decomposed into a translative part and a part which stems only from the rotation we conclude that the energy of a rigid body consists of two parts $E = E_{\text{kin}} + E_{\text{rot}}$. The translative part contains the movement of the center of mass with total mass M and the part for the rotational energy is given by the rotation of a body with moment of inertia Θ with angular velocity ω.
In real applications we no longer deal with discrete mass points rather than with a continuous body. Thus, we make the displacement $m_i \to dm_i$ and $\sum \to \int$, such that

$$\Theta = \int_V \vec{r}^2\,dm\,. \quad (3.90)$$

where the Integral covers the volume V of the considered body. Introducing the **density** $\rho = \frac{dm}{dV}$ this may (a constant density is assumed) be rewritten as

$$\boxed{\Theta = \rho\int_V \vec{r}^2\,dV}\,. \quad (3.91)$$

EXAMPLE

As an example we will calculate the moment of inertia of a cylinder with radius R, total mass m and height h The density thus reads $\rho = \frac{m}{\pi R^2 h}$. Using polar coordinates and $dV = r\,dr\,d\phi$ we obtain

$$\Theta = \frac{m}{\pi R^2 h} \int_0^h dx \int_0^R r dr \int_0^{2\pi} d\phi r^2 = \frac{m}{\pi R^2 h} h \frac{R^4}{4} 2\pi = \frac{1}{2} m r^2 \tag{3.92}$$

Suppose we know the moment of Inertia Θ_{cm} with respect to an axis $\vec{\omega}_{\mathrm{cm}}$ through the center of mass then the **parallel-axis theorem** (Steiner's theorem) allows us to compute the moment of inertia Θ with respect to an axis $\vec{\omega}$, which is parallel to $\vec{\omega}_{\mathrm{cm}}$ but otherwise arbitrary. The distance of the two axis shell be denoted by a. With $\vec{x}_i = \vec{a} + \vec{x}_i'$ we have

$$\sum_i m_i \vec{x}_i^2 = \sum_i m_i (\vec{x}_i')^2 + a^2 \sum_i m_i + 2\vec{a} \sum_i m_i \vec{x}_i' \tag{3.93}$$

Since $\sum_i m_i \vec{x}_i' = 0$ we may write

$$\boxed{\Theta = \Theta_{\mathrm{cm}} + M a^2} \tag{3.94}$$

We now turn to the **angular momentum** of rigid body. For simplicity we assume a rotation about an axis $\vec{\omega}$ through the center of mass $\vec{R}$. Furthermore we assume the axis $\vec{\omega}$ to be a symmetry axis of the body. The angular momentum is given by

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\,. \tag{3.95}$$

We now decompose the vector $\vec{r}$ in a component $\vec{r}_{||}$ parallel to $\vec{\omega}$ and a component $\vec{r}_\perp$ perpendicular to $\vec{\omega}$. We thus have

$$\begin{aligned} \vec{r} \times (\vec{\omega} \times \vec{r}) &= (\vec{r}_\perp + \vec{r}_{||}) \times (\vec{\omega} \times (\vec{r}_\perp + \vec{r}_{||})) = (\vec{r}_\perp + \vec{r}_{||}) \times (\vec{\omega} \times \vec{r}_\perp) \\ &= \vec{r}_\perp \times (\vec{\omega} \times \vec{r}_\perp) + \vec{r}_{||} \times (\vec{\omega} \times \vec{r}_\perp) \\ &= \vec{\omega} r_\perp^2 - \vec{r}_\perp (r_{||}\omega)\,. \end{aligned} \tag{3.96}$$

Since we assume the body to rotate around a symmetry axis it holds $\omega \sum_i m_i r_{||,i} \vec{r}_{\perp,i} = 0$ and we can conclude

$$\vec{L} = \left(\sum_i m_i r_{\perp,i}^2 \right) \vec{\omega} = \Theta \vec{\omega}\,. \tag{3.97}$$

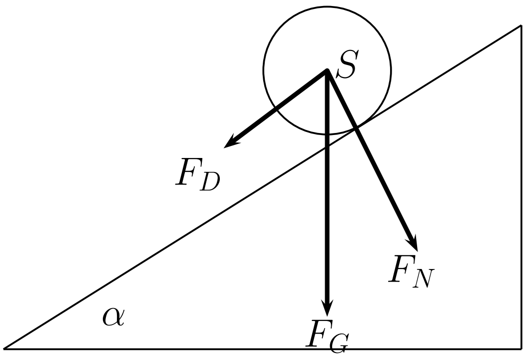


Figure 3.5: A cylinder on an inclined plane.

with the moment of inertia

$$\Theta = \sum_i m_i r_{\perp,i}^2 \,. \tag{3.98}$$

The equation of motion for the rotation of a rigid body is given by

$$\boxed{\dot{\vec{L}} = \vec{M}} \tag{3.99}$$

EXAMPLE

As an example consider a cylinder with radius R and total mass m on an inclined plane as shown in Fig. 3.5. The moment of torque is given by $M = RF_D = mgR\sin\alpha$. The angular momentum is $L = \Theta\omega = (\Theta_{\text{cm}} + mR^2)\omega$. Thus we obtain the equation of motion

$$mgR\sin\alpha = (\Theta_{\text{cm}} + mR^2)\dot{\omega} \quad \Leftrightarrow \quad \dot{\omega} = \frac{mgR\sin\alpha}{\Theta_{\text{cm}} + mR^2} \tag{3.100}$$

The center of mass S is subsequently subject to an acceleration

$$a = R\dot{\omega} = R\frac{mgR\sin\alpha}{\Theta_{\text{cm}} + mR^2} \tag{3.101}$$

At the end of this chapter we list some correspondences between translations and rotations:

Translation	Rotation
Mass m	Moment of inertia Θ
Kinetic energy $\frac{m}{2}v^2$	Rotational energy $\frac{\Theta}{2}\omega^2$
Momentum $\vec{p} = m\vec{v}$	Angular momentum $\vec{L} = \Theta\vec{\omega}$
Force $\vec{F}$	Moment of a torque $\vec{M} = \vec{r} \times \vec{F}$
Equation of motion $\dot{\vec{p}} = \vec{F}$	Equation of motion $\dot{\vec{L}} = \vec{M}$

Chapter 4

Oscillations and waves

4.1 Harmonic motion

We consider a body with mass m in one dimension subject to a force $F = -Dx$. Applying Newton's principle of action ($F = m\ddot{x}$) we get a (second order and linear) differential equation for $x(t)$

$$\boxed{m\ddot{x}(t) = -Dx(t)}\,. \tag{4.1}$$

Its general solution reads

$$\boxed{x(t) = A\sin(\omega\, t + \phi)}\,, \tag{4.2}$$

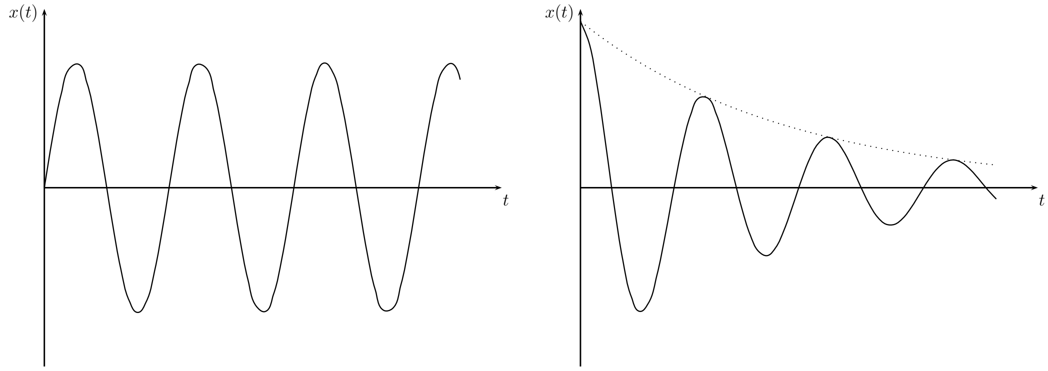
with the **amplitude** A and the **phase** ϕ to be determined by boundary conditions (for instance position and velocity at time $t = 0$).

This kind of motion is called **harmonic oscillation**. It occurs, whenever a mass point is subject to a force, whose magnitude is proportional to amplitude and points in the opposite direction.

(Note: A linear force $F = -Dx$ corresponds to a potential $V(x) = Dx^2$. Whenever a mass point is in a stable position of equilibrium x_0 (i.e. $V'(x_0) = 0$ and $V''(x_0) > 0$) the potential $V(x)$ for small deviations around x_0 may be approximated by a quadratic polynomial. Thus, for small deviations many physical problems can be reduced to a harmonic oscillation.)

If we put equ. (4.2) in equ. (4.1), we get the condition $(m\omega^2 - D)A\sin(\omega\, t + \phi) = 0$ and thus

$$\boxed{\omega = \sqrt{\frac{D}{m}}} \qquad \Leftrightarrow \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{D}}\,. \tag{4.3}$$



(a) (b)

Figure 4.1: (a)Harmonic motion appears when a mass point is subject to a force $F = -Dx$. (b) Damped oscillation: The amplitude decays like $Ae^{-\lambda t}$.

In a real system there will always be some kind of friction. One easy case one might consider is stokes friction. In addition to the reset force $-Dx$ we get a contribution $-\gamma\dot{x}$, such that the differential equation now reads

$$m\ddot{x}(t) + \gamma\dot{x}(t) + Dx(t) = 0\,. \tag{4.4}$$

With the abbreviations

$$\lambda = \frac{\gamma}{2m} \qquad \text{and} \quad \omega = \sqrt{\frac{D}{m} - \frac{\gamma^2}{4m^2}} \tag{4.5}$$

the solution is found to be

$$x(t) = Ae^{-\lambda t}\sin(\omega t + \phi)\,. \tag{4.6}$$

The amplitude of the oscillation decays exponentially with time.

4.2 The mathematical and physical pendulum

A **mathematical pendulum** consists of a mass point m attached to a fiber (assumed to be massless) with length l subject to the gravitational force $F_G = -mg$. We may start from the expression $\dot{L} = M$ with $L = \Theta\dot{\omega} = ml^2\ddot{\alpha}$ and $M = -mg\sin\alpha$. We obtain the following differential equation

$$\ddot{\alpha}(t) = -\frac{g}{l}\sin(\alpha(t))\,. \tag{4.7}$$

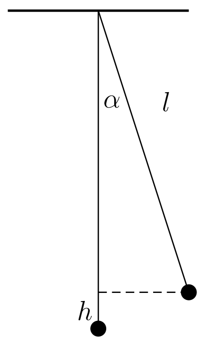


Figure 4.2: The Mathematical pendulum: A mass pony m is attached to a mass-less fiber with length l.

For small angles α we may approximate $\sin(\alpha) \approx \alpha$. Consequently the equation of motion reads

$$\ddot{\alpha}(t) = -\frac{g}{l}\alpha(t) \tag{4.8}$$

and we find the solution to be a harmonic oscillation

$$\alpha(t) = \alpha_0 \sin(\omega t) \tag{4.9}$$

with the angular velocity

$$\omega = \sqrt{\frac{g}{l}}\,. \tag{4.10}$$

In contrast the **physical pendulum** consists of an (arbitrary) body that can rotate about an axis A as shown in Fig. 4.3. The body has mass m and a moment of inertia Θ. S denotes the center of mass. We start from the equation of motion for a rigid body $M = \dot{L} = \Theta\dot{\omega} = \Theta\ddot{\phi}$. The moment of a torque is given by $M = -mgL\sin\phi$, such that

$$-mgL\sin\phi = \Theta\ddot{\phi}\,. \tag{4.11}$$

Again, for small angles $\sin\phi \approx \phi$ and we again obtain the common form for harmonic oscillations

$$\ddot{\phi} + \frac{mgL}{\Theta}\phi = 0\,. \tag{4.12}$$

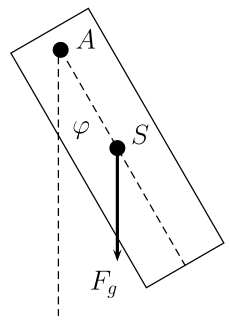


Figure 4.3: Physical pendulum with moment of inertia Θ with respect to the axis A. It is $\vec{AS} = L$.

The solution reads $\phi(t) = \phi_o \sin(\omega t)$ with

$$\omega = \sqrt{\frac{gmL}{\Theta}}\,. \tag{4.13}$$

Comparing this with the formula $\omega = \sqrt{\frac{g}{l}}$ for the mathematical pendulum one may write

$$\omega = \sqrt{\frac{g}{l_r}} \qquad \text{with} \quad l_r = \frac{\Theta}{mL} \tag{4.14}$$

with the so called **reduced length of the pendulum** l_r.

4.3 Enforced Oscillations

We consider now a simple harmonic oscillator (i.e. a mass m attached to a spring with spring constant D) subject to a harmonic external force $F_{\text{ext}} = F_0 \cos(\Omega t)$. The equation of motion thus takes the form

$$m\ddot{x}(t) = -Dx(t) + F_0 \cos(\Omega t) \tag{4.15}$$

or with $\omega^2 = \frac{D}{m}$

$$\ddot{x}(t) + \omega^2 x(t) = \frac{F_0}{m} \cos(\Omega t)\,. \tag{4.16}$$

To solve this equation of motion we make the ansatz:

$$x(t) = A\cos(\Omega t) \tag{4.17}$$

Putting this ansatz in equ. (4.16) we get

$$A(-\Omega^2 + \omega^2) = \frac{F_0}{m} \quad \Leftrightarrow \quad A = \frac{F_0}{m}\frac{1}{\omega^2 - \Omega^2}\,. \tag{4.18}$$

The entire solution can now be simply read off

$$x(t) = \frac{F_0}{m}\frac{1}{\omega^2 - \Omega^2}\cos(\Omega t)\,. \tag{4.19}$$

The solution contains the following features:

(i) The body oscillates with same frequency Ω, which it is forced on by the external force.

(ii) The amplitude A is strongly dependent on the external frequency Ω and grows without bound if Ω reaches the natural frequency ω, which is in this context also called **resonant frequency**.

(iii) Crossing the resonant frequency ω from below ($\Omega < \omega$) involves a change of the sign of A, or in other words a phase shift about π.

The absolute value of A is shown in Fig. 4.4.
In real system there is always friction. We may again consider the simplest

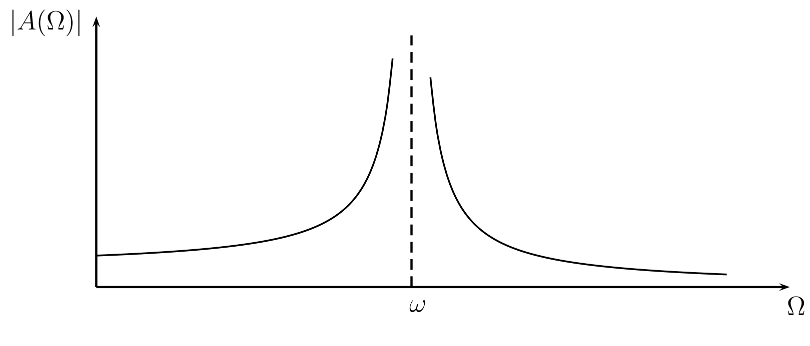


Figure 4.4: Absolute value $|A|$ of the amplitude in the case of an enforced harmonic oscillation.

case of stokes friction. Adding a friction term on the right hand side of equ. (4.15) leads to the equation of motion for a **damped enforced oscillation**:

$$m\ddot{x} + \gamma\dot{x} + Dx = F_0\cos(\Omega t)\,. \tag{4.20}$$

Solving this equation needs more mathematical background and we will content ourself only with the solution

$$x(t) = A\cos(\omega t + \phi)\,. \tag{4.21}$$

where

$$A = \frac{F_0}{\sqrt{m^2(\omega^2-\Omega^2)^2 + D^2\Omega^2}} \qquad \text{and} \qquad \tan\phi = \frac{k\Omega}{m(\omega^2-\Omega^2)}\,. \tag{4.22}$$

Due to the damping the absolute value $|A|$ is now bounded as shown in Fig. 4.5 and the phase shift appears smoothly.

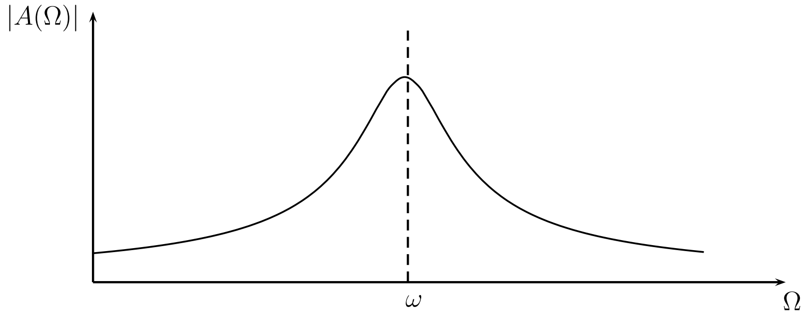


Figure 4.5: Absolute value $|A|$ of the amplitude in the case of an enforced harmonic oscillation with damping.

4.4 Wave motion

In order to study the motion of waves we assume mass points with mass m in $d = 1$ dimension affiliated with each other by springs, each with the same spring constant D. The equation of motion for mass point n reads

$$\begin{aligned} m\ddot{u}_n &= -D(u_n - u_{n-1}) + D(u_{n+1} - u_n) \\ &= D(u_{n+1} - 2u_n + u_{n-1})\,. \end{aligned} \tag{4.23}$$

We are interested in the so called continuum limit, that is $a, D, m \to 0$ with fixed quantities

$$\mu := \frac{m}{a} \qquad \text{and} \qquad Y := aD\,. \tag{4.24}$$

Defining $a := \Delta x$ and $na = x$, we can write

$$\mu \Delta x \ddot{u}_n = \frac{Y}{\Delta x}(u(x+\Delta x) - 2u(x) + u(x-\Delta x)) \tag{4.25}$$

or

$$\ddot{u}(x) = \frac{Y}{\mu}\frac{(u(x+\Delta x) - 2u(x) + u(x-\Delta x))}{(\Delta x)^2} \tag{4.26}$$

If we now take the limit $\Delta x \to 0$ the last fraction turns out be the second derivative of $u(x)$ with respect to x. Thus

$$\ddot{u}(x,t) - \frac{Y}{\mu}u''(x,t) = 0 \tag{4.27}$$

or with the so called **phase velocity** $c = \sqrt{\frac{Y}{\mu}}$

$$\boxed{\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u(x,t) = 0}. \tag{4.28}$$

This equation is called **wave equation**. We will consider one special case of solutions, namely

$$u(x,t) = u_0 \cos(\omega t - kx) \tag{4.29}$$

If we put this ansatz in equ. 4.28 we obtain a dependency of ω and k, that is

$$-k^2 + \frac{\omega^2}{c^2} = 0 \qquad \Leftrightarrow \qquad \boxed{\omega = ck}. \tag{4.30}$$

The latter equation is called **dispersion relation**.

4.5 Superposition and Reflection of waves

The wave equation is a liner equation, which implies that if $u_1(x,t)$ and $u_2(x,t)$ are solutions, than

$$u(x,t) = \alpha u_1(x,t) + \beta u_2(x,t) \tag{4.31}$$

is another solution. We will consider now a fixed position $x = 0$ and the two waves $u_1(0,t) = u_1(t) = \cos(\omega_1 t)$ and $u_2(0,t) = u_2(t) = \cos(\omega_2 t)$. The sum $u(t)$ with the aid of $\cos(a) + \cos(b) = 2\cos(\frac{a-b}{2})\cos(\frac{a+b}{2})$ is found to be

$$u(x,t) = \cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right). \tag{4.32}$$

Fig. 4.6 shows, what happens if two frequencies are added, which are not much distinguishable from each other, i.e. $\omega_1 \approx \omega_2$. The first term describes the slow oscillating envelope with frequency $\omega_e = \left|\frac{\omega_1 - \omega_2}{2}\right|$.

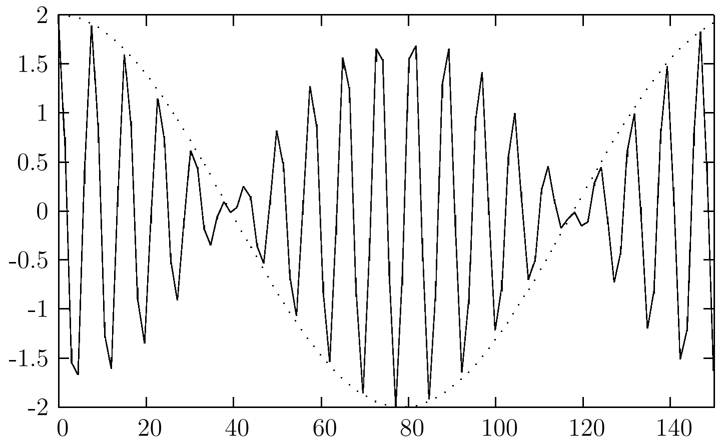


Figure 4.6: Adding two cos-waves with almost the same frequencies leads to beat.

4.6 Doppler effect

Moved source
We assume a sound source traveling with velocity v_s and emitting waves with wavelength $\lambda = 1/f$ towards a detector D. The velocity of the wave is denoted as usual by c. Within one period of oscillation $T = \frac{\lambda}{c}$ the source travels a distance of $v_s T = v_s \frac{\lambda}{c}$. Thus, the detector measures a wavelength

$$\lambda' = \lambda - v\frac{\lambda}{c} = \lambda\left(1 - \frac{v_s}{c}\right) . \tag{4.33}$$

With $1/f = \lambda$ we obtain

$$\boxed{f' = f\frac{1}{1 - \frac{v_s}{c}}} . \tag{4.34}$$

Note, that for $v_c = c$ the frequency f' becomes infinity. This is called sound barrier.
Moved detector
If the detector is moved with velocity v_d and the source stays at rest, than the time T' needed to pass two consecutive maxima is

$$T' = \frac{\lambda}{c + v_d} \tag{4.35}$$

Thus, the frequency $f' = 1/T'$ the detector measures is given by

$$f' = \frac{c}{\lambda}\left(1 + \frac{v_d}{c}\right) \tag{4.36}$$

or with $f = \frac{c}{\lambda}$

$$\boxed{f' = f\left(1 + \frac{v_d}{c}\right)}. \tag{4.37}$$

Part II

Classical Electrodynamics

Chapter 5

Electrostatics

5.1 Introduction

Electric charge is quantised, i.e. every charge Q can be written as

$$Q = n \cdot e, \quad \text{with} \quad n \in \mathbb{Z} \tag{5.1}$$

with the **elementary charge**

$$e = 1.602 \cdot 10^{-19}\, C\,. \tag{5.2}$$

It holds:

(i) Homonymous charges repel each other.

(i) In-homonymous charges attract each other.

5.2 Maxwell's Equations

The fundamental laws in classical mechanics are Newton's axioms (in particular the principle of action). An analogon exists in the field of classical electrodynamics - the four Maxwell equations, which are heuristic equations (i.e. the can not be derived from some "higher law"). We will list these equations in its integral form and explain their matter.

(1) **Gauss' law**
Gauss's law gives the relation between the electric flux flowing out a closed surface and, respectively, the electric charge enclosed in the surface:

$$\int_S \vec{E} d\vec{A} = \frac{Q_{in}}{\epsilon_0}$$

with the **permittivity of free space**

$$\epsilon_0 = 8.85 \cdot 10^{-12} \frac{As}{Vm} \,. \tag{5.3}$$

(2) **Gauss' law for magnetism**
Gauss' law for magnetism merely states the absence of magnetic monopoles:

$$\int_S \vec{B} d\vec{A} = 0$$

(3) **Faraday's law of induction**
Faraday's law of induction (more generally, the law of electromagnetic induction) states that a magnetic field changing in time creates a proportional electro-motive force. The relation between the rate of change of the magnetic flux through the surface S enclosed by a contour C and the electric field along the contour reads:

$$\int_C \vec{E} d\vec{l} = -\frac{d}{dt} \int_S \vec{B} d\vec{A}$$

(4) **Ampere's law**
Ampere's law, discovered by Andree-Marie Ampere, relates the circulating magnetic field in a closed loop to the electric current passing through the loop. It is the magnetic equivalent of Faraday's law of induction.

$$\int_C \vec{B} d\vec{l} = \mu_0 \int_S \vec{j} d\vec{A} + \frac{d}{dt} \int_S \vec{E} d\vec{A}$$

with the **magnetic constant** or the **permeability of vacuum**

$$\mu_0 = 1.26 \cdot 10^{-6} \frac{Vs}{Am} \,. \tag{5.4}$$

5.3 The Electric Field - Coulomb's Law

From Gauss' law the electric field $\vec{E}$ of a point charge q at a distance r can be calculated when considering a sphere centered around that point charge (s. fig. 5.1). Due to the symmetry the electric field is constant there, this gives

$$\int_S \vec{E} d\vec{A} = \vec{E} \cdot 4\pi r^2 \vec{e}_r = \frac{q}{\epsilon_0} \qquad \text{or} \qquad \boxed{E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}} \,. \tag{5.5}$$

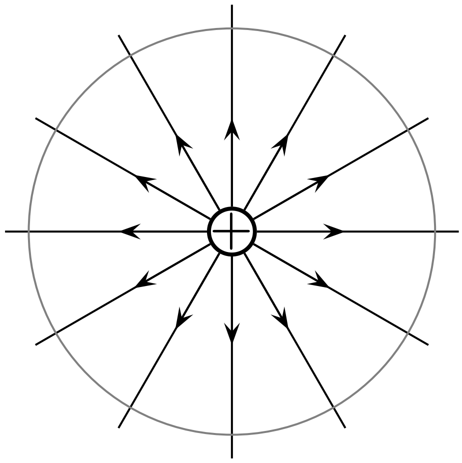


Figure 5.1: Cut through the spherical electric field of a positive point charge and a sphere where the electric field is constant.

Since the **electric field** represents the force per charge, this leads to the force between two elementary charges $q_1 = Z_1 e_1$ and $q_2 = Z_2 e_2$ with $Z_i = \pm 1\, i = 1, 2$ as

$$\vec{F}_{12} = q_2 \vec{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r_1} - \vec{r}_2|^3} (\vec{r_1} - \vec{r}_2) \qquad \text{or} \quad F_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}^2} \tag{5.6}$$

where $r_{12} = |\vec{r_1} - \vec{r}_2|$ states the distance between the two charges. This result, known as Coulomb's law, is in analogy to the gravitational force between two mass points with masses m_1 and m_2.

Now we consider two plates with area A, separated by a distance d, each carrying a charge density $\sigma = \pm \frac{Q}{A}$ (s. fig. 5.2). In order to calculate the electric field between the plates we make again use of the first maxwell equation.

$$\int_S \vec{E} d\vec{A} = EA = \frac{Q}{\epsilon_0} \quad \Leftrightarrow \quad E = \frac{\sigma}{\epsilon_0}\,. \tag{5.7}$$

Thus, we obtain a homogeneous (i.e. constant in space) electric field which corresponds to a homogeneous force

$$F = qE = q\frac{\sigma}{\epsilon_0} \tag{5.8}$$

and therefore a motion with constant acceleration.

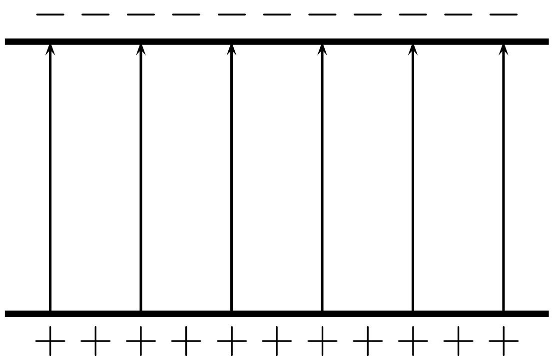


Figure 5.2: Homogeneous Electric Field between two plates with charge $\pm Q$, separated by a distance d.

5.4 Electric Potential, Voltage and Work in the Electric Field

When describing an electric force it is often easier to use the scalar **electric potential** ϕ instead of the vector of the electric field. It is given by

$$\phi(\vec{x}) = -\int \vec{E} d\vec{x} \,. \tag{5.9}$$

In applications, however, the derived quantity **Voltage** is used, which is defined by the difference of the potential at two points $\vec{x}_1$ and $\vec{x}_2$:

$$U = \Delta\phi = \phi(\vec{x}_2) - \phi(\vec{x}_1) \qquad [U] = V \tag{5.10}$$

As an example, reconsider the two plates in fig. 5.2. Integrating the constant electric field $E = \frac{\sigma}{\epsilon_0}$ yields

$$\int_0^d E dx = Ed = U \quad \Leftrightarrow \quad \boxed{E = \frac{U}{d}} \qquad [E] = \frac{V}{m} \,. \tag{5.11}$$

Therefore the electric field between two parallel plates can be expressed as a ratio of the voltage between them to their distance d. With this result the work of an electric field given by the general formula

$$W = \int \vec{F} d\vec{x} = q \int \vec{E} d\vec{x} = qU \tag{5.12}$$

which is done when an electric charge is transported from one plate to the other can simply be expressed in case of a homogeneous field as

$$W = q \int_0^d E\, dx = qEd\,. \tag{5.13}$$

5.5 Motion in a Homogeneous Electric Field

We consider a particle of mass m and charge q, which is accelerated due to a (acceleration) voltage U. During this process the particle gains kinetic energy, i.e.

$$\boxed{qU = \frac{1}{2}mv^2} \quad \Leftrightarrow \quad v = \sqrt{\frac{2qU}{m}}\,. \tag{5.14}$$

Afterwards the particle enters a parallel plate capacitor as depicted in Fig. 5.3. Due to the the electric field the particle is accelerated only in y-direction. With $a = \frac{F}{m} = \frac{qE}{m} = \frac{qU}{md}$ the motion is described as

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} vt \\ \frac{qU}{2md}t^2 \end{pmatrix}\,. \tag{5.15}$$

If we are only interested in the geometrical path we can solve for $x = vt \Leftrightarrow$

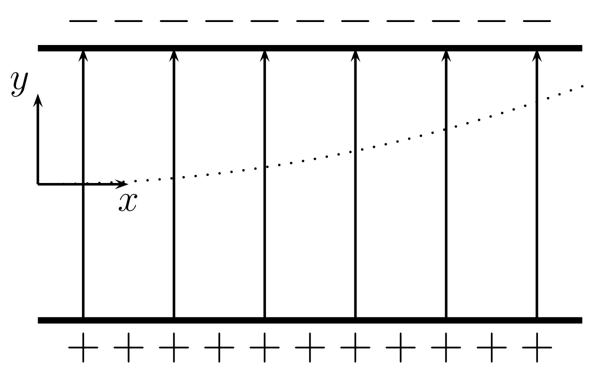


Figure 5.3: If a charged particle enters a parallel plate capacitor it is accelerated in the y-direction. The particle's path forms a parabola.

$t = x/v$. If we put this in $y(t)$ we get

$$\boxed{y(x) = \frac{1}{2}\frac{qU}{mdv^2}x^2}\,. \tag{5.16}$$

5.6 Capacity

The **capacity** is defined as the quotient of charge and voltage:

$$\boxed{C = \frac{Q}{U}} \qquad [C] = F \tag{5.17}$$

In the case of the two parallel plates, we obtain

$$C = \frac{Q}{U} = \frac{\sigma A}{Ed} = \epsilon_0 \frac{A}{d}\,. \tag{5.18}$$

With the capacity the work which is stored inside a parallel plate capacitor can be expressed by

$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CU^2\,. \tag{5.19}$$

For a parallel plane capacitor this can be expressed by the electric field:

$$W = \frac{1}{2}\frac{A}{d}(dE)^2 = \frac{1}{2}V\epsilon_0 E^2 \tag{5.20}$$

This means their density is propotional to the electric field squard, which can be derivated generally, too.

5.7 Matter in an Electric Field

We reconsider the parallel plate capacitor. Putting a dielectric material between the plates causes an increase in the capacitance in proportion to ϵ_r, the relative permittivity of the material:

$$\boxed{C = \epsilon_r \epsilon_0 \frac{A}{d}}\,. \tag{5.21}$$

This happens because an electric field polarises the molecules of the dielectric, producing concentrations of charge on its surfaces that create an electric field opposed (anti-parallel) to that of the capacitor. Thus, a given amount of charge produces a weaker field between the plates than it would without the dielectric, which reduces the electric potential. Considered in reverse, this argument means that, with a dielectric, a given electric potential causes the capacitor to accumulate a larger charge.

Chapter 6

Electric Direct Current

6.1 Electric Current

As a simple model for the electric current we consider a conductor, which has a density of charge carriers (i.e. electrons with mass $m = m_e$) n. We want to write down the equation of motion for one of this carriers if it is exposed to an electric field E. This particular electron will make collisions with all the other electrons. The time between two collisions will be denoted by τ. These collision processes effectively lead to a friction force, which we will model as Stokes's friction $\gamma\, v$. Thus, the equation of motion takes the form

$$m\dot{v}(t) = eE - \frac{m}{\tau}v(t)\,. \tag{6.1}$$

A static (time independent) solution of this problem is given by

$$v = \frac{e\tau}{m}E\,. \tag{6.2}$$

This is the velocity of one carrier. The total velocity of all carriers is $v_{tot} = n\,v$. Thus the electric current density reads

$$j = ev_{tot} = \frac{ne^2\tau}{m}E = \sigma\, E \tag{6.3}$$

where we defined the **conductivity**

$$\sigma = \frac{ne^2\tau}{m}\,. \tag{6.4}$$

If we are dealing with a homogeneous conductor with length L and cross section A equation 6.4 may be integrated to yield:

$$\int jdA = jA = I \quad \text{and} \quad \int Edl = EL = U \tag{6.5}$$

Thus

$$\boxed{U = \frac{L}{\sigma A} I = R\, I}. \qquad [I] = A \tag{6.6}$$

This relation is known as (the famous) **Ohm's law**. We defined the **resistivity**

$$R = \frac{1}{\sigma}\frac{L}{A} = \rho_s \frac{L}{A} \qquad [R] = \Omega \tag{6.7}$$

with the so called **specific resistance** $\rho_s = \frac{1}{\sigma}$.

6.2 Kirchhoff's rules

In electric circuits one often deals with several connected conductors. To calculate for instance the individual currents and voltages or the total resistivity of the system **Kirchhoff's rules** provide a very useful tool.

(i) **Kirchhoff's current law**
At any point in an electrical circuit where charge density is not changing in time, the sum of currents flowing towards that point is equal to the sum of currents flowing away from that point:

$$\sum_k I_k = 0 \tag{6.8}$$

(ii) **Kirchhoff's voltage law**
From energy conservation follows, that the directed sum of the electrical potential differences around a circuit must be zero:

$$\sum_k U_k = 0 \tag{6.9}$$

From there one can e.g. find the rules for the series connection and the parallel connection of resistors. One finds

(ii) **Series connection**

$$R = \sum_k R_k \tag{6.10}$$

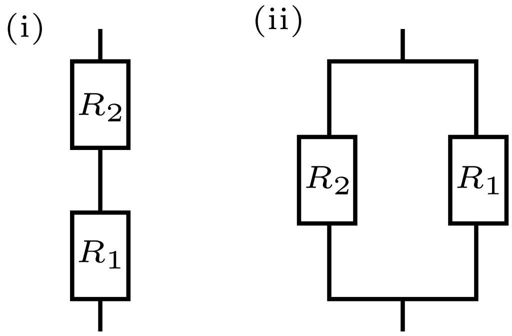


Figure 6.1: Series (i) and parallel (ii) connection of two resistors.

(ii) **Parallel connection**

$$R = \frac{1}{\sum_k \frac{1}{R_k}} \tag{6.11}$$

For capacitances the following rules hold

(ii) **Series connection**

$$C = \frac{1}{\sum_k \frac{1}{C_k}} \tag{6.12}$$

(ii) **Parallel connection**

$$C = \sum_k C_k \tag{6.13}$$

6.3 Electric work and power

We consider a again a conductor with resistance R and an applied voltage U through which flows a current I. In it energy is dissipated, given by

$$P = \frac{dW}{dt} = U\frac{dQ}{dt} = UI\,. \tag{6.14}$$

Taking into account Ohm's law, i.e $U = RI$ this may be rewritten as

$$\boxed{P = UI = \frac{U^2}{R}}. \qquad [P] = W \tag{6.15}$$

Chapter 7

Electromagnetism

7.1 The Magnetic field

Consider a long wire which carries a current I. By reason of symmetry and on the basis of the second maxwell equation $\int_S \vec{B} d\vec{A} = 0$ the magnetic field has only a tangential component. This can be calculated by the fourth Maxwell equation (Ampere's law):

$$\int_C \vec{B} d\vec{l} = B\, 2\pi r = \mu_0 \int_S \vec{j} d\vec{A} = \mu_0 I \quad \Leftrightarrow \quad B = \frac{\mu_0 I}{2\pi r} \tag{7.1}$$

In order to find the magnetic field of a coil with length L and number of turns N we again make use of Ampere's law. The right hand side of this equation yields BL and for the right hand side we get $\mu_0 N I$. Thus, the magnetic field of a coil is given by

$$\boxed{B = \mu_0 \frac{N}{L} I}. \tag{7.2}$$

Thus, we obtain in a coil a homogeneous magnetic field.

7.5 Lorentz Force

The **Lorentz Force** is the force exerted on a charged particle in an electromagnetic field. The particle will experience a force due to electric field of $q\vec{E}$, and due to the magnetic field $q\vec{v} \times \vec{B}$. Combined they give the Lorentz force equation (or law):

$$\boxed{\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B})} \tag{7.3}$$

In this chapter we will consider the second term only (due to the magnetic field $\vec{B}$). If the magnetic field is perpendicular to the velocity we have $F_L = qvB$. Since a charged particle is moving on a circle in such a magnetic field we can equate the Lorentz force and the centrifugal force to obtain

$$qvB = \frac{mv^2}{r} \quad \Leftrightarrow \quad v = \frac{qBr}{m} \,. \tag{7.4}$$

Next we want to calculate the force F on a wire of length l, which carries a current I. For a magnetic field B perpendicular to the direction of the current we obtain

$$F = IlB \,.$$

The velocity selector
Consider a region of space between the plates of a capacitor where there is an electric field and a perpendicular magnetic field. A particle of charge q enters this space from the left. If q is positive, the electric force of magnitude qE points down and the magnetic force of magnitude qvB points up. If the charge is negative, each of these two forces is reversed. The forces balance at

$$qE = qvB \quad \Leftrightarrow \quad v = \frac{E}{B} \,. \tag{7.5}$$

7.3 Magnetic flux and magnetic induction

In order to characterise the magnetic field by one scalar value, the magnetic flux Φ is defined via

$$\Phi = \int \vec{B} d\vec{A} = BA \tag{7.6}$$

where the last equality holds only for a homogeneous field. Thus, Maxwell's third equation (Faraday's law of induction) may be rewritten as

$$\int_C \vec{E} d\vec{l} = -\frac{d}{dt} \int_S \vec{B} d\vec{A} = -\dot{\Phi} \,. \tag{7.7}$$

Since the left hand side of this equations is a voltage we finally have

$$\boxed{U_{\text{ind}} = -\dot{\Phi}} \,. \tag{7.8}$$

7.4 Self-inductance

We consider a coil with

$$B = \mu_0 I \frac{N}{L}\,. \tag{7.9}$$

If the current I through the coil changes with time we get an induced voltage

$$U_{\text{ind}} = -\frac{d}{dt}(BA)N = -\frac{d}{dt}\left(A\mu_0 I \frac{N}{L} N\right) = -\left(\mu_0 \frac{AN^2}{L}\right)\dot{I}\,. \tag{7.10}$$

If we define the **inductance** to be

$$L = \mu_0 \frac{AN^2}{L} \tag{7.11}$$

we get

$$\boxed{U_{\text{ind}} = -L\dot{I}}\,. \tag{7.12}$$

The last equation is **Lenz's law**:

The induced current produced in the conductor always flows in such a direction that the magnetic field it produces will oppose the change that produces it.

7.6 Electromagnetic machines

In this section we will consider some important examples for application, namely the electric motor, electric generator and the transformer.

Electric generator
We consider a rectangular coil of N turns and with area A in a uniform magnetic field B. In the simplest picture of a generator this rectangular coil is now (due to an external force) rotated with angular velocity ω. Thus, the effective flux through the rectangular coil is $\Phi = BA\cos(\omega t)$ and we get for the induced voltage

$$U(t) = -\dot{\Phi} = BA\omega\sin(\omega t)\,. \tag{7.13}$$

Thus the generator converts mechanical energy into electric energy.

Electric motor
To convert electric energy into mechanical energy we have to apply an external a.c. voltage to the coil, i.e.

$$U_{\text{ext}} = U_0 \sin(\omega t)\,. \tag{7.14}$$

Due to that external a.c. voltage the coil will rotate with angular velocity ω.

Transformer
If a time-varying voltage U_P is applied to the primary winding of N_P turns, a current will flow in it producing a magneto-motive force. Just as an electromotive force drives current around an electric circuit, so magneto-motive force tries to drive magnetic flux through a magnetic circuit. The primary magneto-motive force produces a varying magnetic flux Φ_P in the core, and, with an open circuit secondary winding, induces a back electro-motive force in opposition to U_P. In accordance with Faraday's law of induction, the voltage induced across the primary winding is proportional to the rate of change of flux:

$$U_P = N_P \frac{d\Phi_P}{dt} \quad \text{and} \quad U_S = N_S \frac{d\Phi_S}{dt} \tag{7.15}$$

where

- U_P and U_S are the voltages across the primary winding and secondary winding.
- N_P and N_S are the numbers of turns in the primary winding and secondary winding.
- $d\Phi_P/dt$ and $d\Phi_S/dt$ are the derivatives of the flux with respect to time of the primary and secondary winding.

Saying that the primary and secondary windings are perfectly coupled is equivalent to saying that $\Phi_P = \Phi_S$. Substituting and solving for the voltages shows that:

$$\frac{U_P}{U_S} = \frac{N_P}{N_S} \tag{7.16}$$

Hence in an ideal transformer, the ratio of the primary and secondary voltages is equal to the ratio of the number of turns in their windings, or alternatively, the voltage per turn is the same for both windings. The ratio of the currents in the primary and secondary circuits is inversely proportional to the turns ratio. This leads to the most common use of the transformer: to convert electrical energy at one voltage to energy at a different voltage by means of windings with different numbers of turns. In a practical transformer, the higher-voltage winding will have more turns, of smaller conductor cross-section, than the lower-voltage windings.

Chapter 8

Electric Oscillations and Electric Waves

8.1 Electric Resonant Circuits

An RLC circuit (also known as a resonant circuit or a tuned circuit) is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel. We will consider only the series connection. Since the total voltage amounts to be zero and with $I = \dot{Q}$ we obtain

$$0 = \frac{Q}{C} + RI + L\dot{I} = \frac{1}{C}Q + R\dot{Q} + L\ddot{Q}\,. \tag{8.1}$$

This equation is analogous to the mechanical equation for damped oscillations $Dx + \gamma\dot{x} + m\ddot{x} = 0$ if one makes the following replacements:

$$x \leftrightarrow Q \qquad m \leftrightarrow L \qquad \gamma \leftrightarrow R \qquad D \leftrightarrow \frac{1}{C} \tag{8.2}$$

We will first consider the case $R = 0$. Eq. (8.1) simplifies to

$$\frac{1}{C}Q(t) + L\ddot{Q}(t) = 0\,. \tag{8.3}$$

This differential equation for $Q(t)$ has the solution

$$Q(t) = Q_0 \sin(\omega t + \phi_0) \qquad \text{with} \quad \omega = \frac{1}{\sqrt{LC}}\,. \tag{8.4}$$

For the general case (i.e. $R \neq 0$) we can take once the time derivative of eq. (8.1) to obtain

$$0 = \frac{1}{C}I(t) + R\dot{I}(t) + L\ddot{I}(t)\,. \tag{8.5}$$

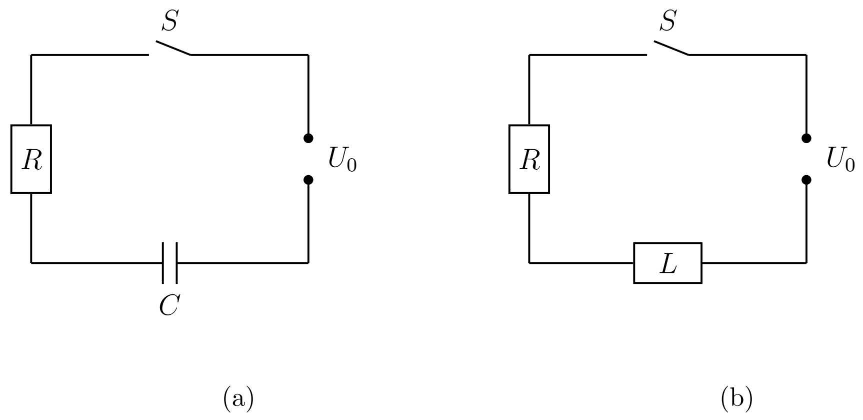


Figure 8.1: (a) RC circuit and (b) RL circuit.

The solution (for $R^2 < \frac{4L}{C}$) is given by

$$I(t) = Ae^{-\gamma t}\cos(\omega t + \phi) \quad \text{with} \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}, \quad \gamma = \frac{R}{2L}\,. \tag{8.6}$$

8.2 Opening and Closing of a Circuit

In this section we will consider the **opening and closing** of a **RC** and a **RL** circuit (s. fig. 8.1).

RC-Circuit

(i) **Closing of the RC Circuit**
We assume that the switch S is closed at time $t = 0$. From Kirchhoff's voltage law we get

$$U_0 = R\dot{Q}(t) + \frac{1}{C}Q(t) \tag{8.7}$$

which solution with $Q(0) = 0$ is given by

$$Q(t) = CU_0\left(1 - e^{-\frac{t}{RC}}\right)\,. \tag{8.8}$$

(ii) **Opening of the RC Circuit**
If the switch is opened at time $t = 0$ (i.e. $U_0 = 0$) we have with $I = -\dot{Q}$:

$$\frac{1}{C}Q(t) + R\dot{Q}(t) = 0 \tag{8.9}$$

The solution to this homogen differential equation with $Q(0) = Q_0$ is given by

$$Q(t) = Q_0 e^{-\frac{t}{RC}}\,. \tag{8.10}$$

RL-Circuit

(i) **Closing of the RL Circuit**
We assume that the switch S is closed at time $t = 0$. Kirchhoff's voltage law states for this case

$$U_0 = I(t)R + L\dot{I}(t) \tag{8.11}$$

with the solution for $I(0) = 0$:

$$I(t) = \frac{U_0}{R}\left(1 - e^{-\frac{R}{L}t}\right) \tag{8.12}$$

(ii) **Opening of the RL Circuit**
If the switch is opened at time $t = 0$ (i.e. $U_0 = 0$) we have

$$0 = I(t)R + L\dot{I}(t)\,. \tag{8.13}$$

The solution with $I(0) = I_0$ is given by

$$I(t) = I_0 e^{-\frac{R}{L}t}\,. \tag{8.14}$$