Physics Course - Solutions   Summer 2009

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1. Exercise: The tuned circuit

A capacitor with $C = 2\,\mu$F is put in series to an inductor with $L = 6\,\mu$H and a resistor with $R = 1\,\Omega$.

(i) Draw the circuit.

(ii) What kind of functional behaviour will perform the current?

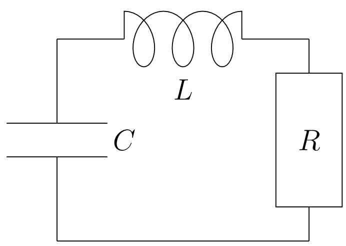
(iii) Calculate the (possible) period of the oscillation?

(iv) When will the current I_0 possesses its maximum value and how large is it, if initially the current vanishes and the capacitor is charged by $Q_0 = 2\,\mu$C?

(v) Determine the power $P_R(t)$ which is dissipated in the resistor.

Solution:

(i)



(ii) Since the resistance is small

$$1.0\,\Omega = R < \sqrt{\frac{4L}{C}} = 3.5\,\Omega$$

the frequency $\Omega = \omega\sqrt{1 - \frac{R^2C}{4L}}$ with $\omega = \frac{1}{\sqrt{LC}}$ is well defined. Thus the current will perform a $\underline{\underline{\text{damped oscillation}}}$.

(iii) The period of that oscillation is given by

$$\underline{\underline{T}} = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\omega^2 - \gamma^2}} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} = \underline{\underline{23\,\mu\text{s}}}$$

(iv) The maximal value of the current is obtained after one quarter of the period:

$$\underline{\underline{t_{1/4}}} = \frac{T}{4} = \underline{\underline{5.7\,\mu\text{s}}}$$

Using the trigonometric formula

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \Rightarrow \qquad \sin(\arcsin a + \beta) &= a\left(\cos\beta + \sqrt{\frac{1}{a^2}-1}\,\sin\beta\right) \end{aligned}$$

the current can be derived by the negative derivation of the solution of the charge:

$$\begin{aligned} I_0(t) &= -\dot{Q} \\ &= -\frac{\mathrm{d}}{\mathrm{d}t} A\mathrm{e}^{-\gamma t}\cos(\Omega t+\phi) \\ &= A\gamma\mathrm{e}^{-\gamma t}\cos(\Omega t+\phi) + A\Omega\mathrm{e}^{-\gamma t}\sin(\Omega t+\phi) \\ &= A\sqrt{\gamma^2+\Omega^2}\mathrm{e}^{-\gamma t}\frac{\gamma}{\sqrt{\gamma^2+\Omega^2}}\left(\cos(\Omega t+\phi) + \frac{\Omega}{\gamma}\mathrm{e}^{-\gamma t}\sin(\Omega t+\phi)\right) \\ &= A\sqrt{\gamma^2+\Omega^2}\mathrm{e}^{-\gamma t}\sin\left(\Omega t+\phi+\arcsin\frac{\gamma}{\sqrt{\gamma^2+\Omega^2}}\right) \\ &= A\omega\mathrm{e}^{-\gamma t}\sin\left(\Omega t+\phi+\arcsin\frac{\gamma}{\omega}\right) \end{aligned}$$

Since the initial current should vanish the sinus in this last expression have to be zero:

$$I_0(t=0\,\text{s}) = 0 \qquad \Rightarrow \qquad \sin\left(\Omega t+\phi+\arcsin\frac{\gamma}{\omega}\right) \qquad \Rightarrow \qquad \phi = -\arcsin\frac{\gamma}{\omega}$$

Putting this in the solution of the charge the remaining parameter can be obtained by the condition that initial the charge have to be Q_0:

$$\begin{aligned} Q_0 &= Q(t=0\,\text{s}) = A\cos\arcsin\frac{\gamma}{\omega} \\ &= A\sqrt{1-\frac{\gamma^2}{\omega^2}} = A\frac{\sqrt{\omega^2-\gamma^2}}{\omega} \\ &= A\frac{\Omega}{\omega} \\ \Rightarrow \qquad A &= \frac{\omega}{\Omega}Q_0 \end{aligned}$$

Since at the maximum current the sinus function in the formula of the current becomes unity the maximum current will be

$$\underline{\underline{I_{\text{max}}}} = \frac{\omega^2}{\Omega}Q_0\mathrm{e}^{-\gamma\frac{T}{4}} = \frac{Q_0}{\sqrt{LC-\frac{R^2C^2}{4}}}\mathrm{e}^{-\frac{R}{2L}\frac{T}{4}} = \underline{\underline{376\,\text{mA}}}$$

(v) Determining the voltage from Ohm's law $R = \frac{U_R(t)}{I(t)}$, the power can be calculated:

$$\begin{aligned} \underline{\underline{P_R(t)}} = U_R(t)I(t) = RI(t)^2 &= \underline{\underline{\tilde{A}\mathrm{e}^{-\tilde{\gamma}}\sin^2(\Omega t)}} \\ \text{with} \quad \underline{\tilde{A}} &= \frac{RQ_0^2}{LC-\frac{R^2C^2}{4}} = \underline{141\,\text{mW}} \\ \text{and} \quad \underline{\tilde{\gamma}} &= \frac{R}{L} = \underline{167\,\text{kHz}} \qquad \underline{(\Omega = 276\,\text{kHz})} \end{aligned}$$

2. Exercise: A capacitor as battery

A charged parallel plate capacitor with area $A = 50\,\mathrm{cm}^2$, distance $d = 1\,\mu\mathrm{m}$ and a material with $\epsilon_r = 40$ between the plates is used as voltage source for two parallel connected resistors with resistances $R_1 = 1.5\,\mathrm{k}\Omega$ and $R_2 = 3\,\mathrm{k}\Omega$. Initially the capacitor should deliver a voltage of $U_0 = 3\,\mathrm{V}$.

(i) Draw the circuit.

(ii) Which resistance enters in the time dependent part of the current?

(iii) Calculate the time constant τ.

(iv) What charge Q_0 has to be placed on the capacitor in order to get the initial voltage?

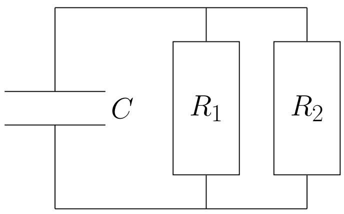
(v) State the initial current $I_{2,0}$ through R_2.

(vi) What is the charge on the capacitor after $t_4 = 4\,\mathrm{ms}$?

(vii) How long does the current at R_2 stay larger than $I_c = 0.1\,\mathrm{mA}$?

Solution:

(i)



(ii) The resistance at the capacitor and therefore the one entering the time constant amounts to

$$\underline{\underline{R}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \underline{\underline{1.0\,\mathrm{k}\Omega}}\,.$$

(iii) The time constant is determined by

$$\underline{\underline{\tau}} = RC = R\epsilon_0\epsilon_r\frac{A}{d} = \underline{\underline{1.8\,\mathrm{ms}}}\,.$$

(iv) The charge on the capacitor is given by

$$C = \frac{Q_0}{U_0} \qquad \Rightarrow \qquad \underline{\underline{Q_0}} = CU_0 = \epsilon_0\epsilon_r\frac{A}{d}U_0 = \underline{\underline{5.3\,\mu\mathrm{C}}}\,.$$

(v) The initial current is given by Ohm's law:

$$R_2 = \frac{U_0}{I_{2,0}} \qquad \Rightarrow \qquad \underline{\underline{I_{2,0}}} = \frac{U_0}{R_2} = \underline{\underline{1.0\,\mathrm{mA}}}$$

(vi) The charge after the time t_4 is given by

$$\underline{\underline{Q(t)}} = Q_0 \mathrm{e}^{-\frac{t_4}{\tau}} = \underline{\underline{0.55\,\mu\mathrm{C}}}$$

(vii) The current can be derived by the derivation of the charge:

$$I_2(t) = -\dot{Q}_2(t) = I_{2,0}\mathrm{e}^{-\frac{t}{\tau}} \qquad \left(\text{with } I_{2,0} = \frac{Q_{2,0}}{\tau} = \frac{Q_{2,0}}{RC}\right)$$

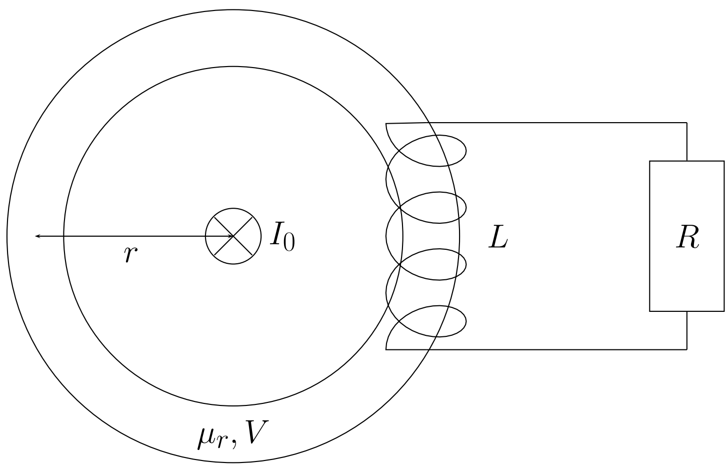
This quantity will be larger than I_c until

$$I_c = I_{2,0}\mathrm{e}^{-\frac{t_c}{\tau}} \qquad \Rightarrow \qquad \underline{\underline{t_c}} = \tau \ln \frac{I_{2,0}}{I_c} = \underline{\underline{4.1\,\mathrm{ms}}}$$

3. Exercise: Residual-current device

Background: Residual-current devices are nowadays used in every household. They provide a recoverable and easy to use safety unit for the customers. Such devices consists of a ferromagnetic ring with the power wire on its axis, to enhance its magnetic field. Around the ring a coil is constructed to get a response if the magnetic field collapses or increases rapidly. The resulting current triggers a relay which disconnects the power wire.

We model the relay as a resistor with $R = 15\,\Omega$ and the connected coil around the ring should have a self inductance of $L = 5\,\mathrm{mH}$. At $t = 0\,\mathrm{s}$ the former current in the power wire of $I_0 = 1\,\mathrm{A}$ is collapsing instantaneous. Take a relative permeability of $\mu_r = 1000$ for the ferromagnetic ring with radius $r = 10\,\mathrm{cm}$ and volume $V = 100\,\mathrm{cm}^3$ centred around the power wire.



(i) Determine the magnetic field B_0 at the position of the coil.

(ii) Calculate the time when the current has decayed to half of its former value $\hat{I}$.

(iii) When does it reach the value $\frac{\hat{I}}{\mathrm{e}}$?

(iv) How much energy is dissipated in the resistor depending on the initial current $\hat{I}$?

(v) What is the magnetic energy stored in the coil related to the current and its self inductance?

(vi) Use the formulas which connect the magnetic field and the self-induction to the geometric quantities of the filled coil to derive a formula stating how the energy stored in the coil is related to the magnetic field inside.

(vii) Calculate from it the initial current $\hat{I}$ at the relay and the dissipated energy assuming that at $t = 0\,\text{s}$ all the magnetic energy is stored in the ferromagnetic ring.

Solution:

(i) The magnetic ring is enhanced by the relative permeability to

$$\underline{\underline{B_0}} = \mu_r B_{\text{PW}}(r) = \mu_r \mu_0 \frac{I_0}{2\pi r} = \underline{\underline{2.0\,\text{mT}}}$$

(ii) The half-value period can be evaluated as

$$\frac{\hat{I}}{2} = \hat{I} e^{-\frac{R}{L} t_{1/2}} \quad \Leftrightarrow \quad \underline{\underline{t_{1/2}}} = \frac{L}{R} \ln \frac{\hat{I}}{\hat{I}/2} = \frac{L}{R} \ln 2 = \underline{\underline{0.23\,\text{ms}}}\,.$$

(iii) The time constant is slight higher than the half-value period and is given by

$$\underline{\underline{\tau}} = \frac{L}{R} \ln \text{e} = \frac{L}{R} = \underline{\underline{0.33\,\text{ms}}}$$

(iv) Due to the time dependent current the voltage will be time dependent, too. Keeping this in mind, the energy can be calculated by integrating the time dependent power:

$$\begin{aligned}
\underline{\underline{W_R}} &= \int_0^\infty P(t)\text{d}t = \int_0^\infty U_R(t) I(t) \text{d}t = \int_0^\infty R I(t)^2 \text{d}t = R\hat{I}^2 \int_0^\infty \text{e}^{-2\frac{R}{L}t} \text{d}t \\
&= R\hat{I}^2 \left(-\frac{L}{2R} \cdot 0 - \left(-\frac{L}{2R} \right) \text{e}^0 \right) = \underline{\underline{\frac{1}{2} L \hat{I}^2 = 2.5\,\text{mH} \cdot \hat{I}^2}}
\end{aligned}$$

(v) The energy conservation law states that the energy dissipated in the resistor has to be the one stored in the coil before the decay:

$$\underline{\underline{W_L}} = W_R = \underline{\underline{\frac{1}{2} L \hat{I}^2}}$$

Since the special geometry of the coil is not necessary to derive this result this formula will be valid for every inductive element.

(vi) From $B = \mu_0 \mu_r \frac{N}{\ell} I$ the current necessary for the magnetic field can be obtained to $I = \frac{B\ell}{\mu_0 \mu_r N}$ which leads to

$$\underline{\underline{W_B}} = W_L = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 \mu_r \frac{N^2 A}{\ell} \cdot \frac{B^2 \ell^2}{\mu_0^2 \mu_r^2 N^2} = \underline{\underline{\frac{1}{2} \frac{V}{\mu_0 \mu_r} B^2}}$$

(vii) From Lenz law the coil has to generate at $t = 0\,\text{s}$ the magnetic field in the ring. Thus from energy conservation the current can be calculated:

$$W_B = W_L \qquad \Rightarrow \qquad \frac{1}{2}\frac{V}{\mu_0\mu_r}B_0^2 = \frac{1}{2}L\hat{I}^2 \qquad \Rightarrow \qquad \underline{\underline{\hat{I}}} = \frac{B_0\sqrt{V}}{\sqrt{\mu_0\mu_r L}} = \underline{\underline{8.0\,\text{mA}}}$$

With this value the energy dissipated is given by

$$\underline{\underline{W_R}} = \frac{1}{2}L\hat{I}^2 = \underline{\underline{0.16\,\mu\text{J}}}$$

Note:

- Permittivity of free space: $\epsilon_0 = 8.85 \cdot 10^{-12}\,\frac{\text{C}}{\text{Vm}}$
- Permeability of vacuum: $\mu_0 = 1.26 \cdot 10^{-6}\,\frac{\text{Vs}}{\text{Am}}$