Classical Mechanics and Elektrodynamics

$\vec{F} = m\ddot{\vec{x}}$

Summer 2011

$\int \vec{E}\mathrm{d}\vec{A} = \frac{Q}{\epsilon_0}$

Dipl.-Phys. Stefan Kremer

27. July 2011

Contents

Chapter 1

Mathematical Preface

1.1 Calculus

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The **derivative** of f is defined by

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \,. \tag{1.1}$$

For calculations it is useful to memorize some functions and their derivatives:

Function $f(x)$	Derivative $f'(x)$
c	0
x^n	nx^{n-1}
e^x	e^x
$\log(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

The derivatives of compositions can be evaluated by the following rules:

Rule	Function $f(x)$	$f'(x)$
Product	$g(x)h(x)$	$g'(x)h(x) + g(x)h'(x)$
Chain	$g\big(h(x)\big)$	$g'\big(h(x)\big) \cdot h'(x)$
Quotient	$\frac{g(x)}{h(x)}$	$\frac{g'(x)h(x)-g(x)h'(x)}{h^2(x)}$

The **anti-derivative** $F(x)$ of a function $f(x)$ is defined by

$$F'(x) = f(x) \qquad \text{or} \qquad \int f(x)\mathrm{d}x = F(x)\,. \tag{1.2}$$

Note: $F(x)$ is not uniquely defined. If $F(x)$ is an anti-derivative of $f(x)$ then $\tilde{F}(x) = F(x) + c$ with $c \in \mathbb{R}$ is another one.

The **definite integral** which represent the area enclosed by the integrand $f(x)$, the x-coordinate and the lines with $x = a$ and $x = b$, is given by

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)\,. \tag{1.3}$$

For the composition of functions the following formulas have proven useful:

Rule	Function $f(x)$	$F(x) = \int f(x)\mathrm{d}x$
Integration by parts	$g(x)h'(x)$	$g(x)h(x) - \int g'(x)h(x)\mathrm{d}x$
Substitution	$g\big(h(x)\big)h'(x)$	$G\big(h(x)\big) = \int g(h)\mathrm{d}h$

1.2 Vector Algebra

A vector $\vec{x} \in \mathbb{R}^n$ can be characterized by its Cartesian coordinates $x_1, \ldots x_n$:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\,, \tag{1.4}$$

It can be added to another one $\vec{y}$ or might be multiplied by a scalar α:

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + x_n \end{pmatrix} \qquad \alpha\vec{x} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix} \tag{1.5}$$

Furthermore its **length** is defined by its absolute value

$$|\vec{x}| = x = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + \ldots x_n^2} \tag{1.6}$$

and the angle ϕ which is enclosed by the vectors $\vec{x}$ and $\vec{y}$, is given by the **dot product** (scalar product)

$$\vec{x} \cdot \vec{y} = \vec{x}\vec{y} = \sum_{i=1}^{n} x_i y_i = xy\cos(\phi) = \vec{y} \cdot \vec{x}\,. \tag{1.7}$$

Therefore the dot product vanishes $\vec{x} \cdot \vec{y} = 0$, if the vectors are perpendicular $\vec{x} \perp \vec{y}$.

Additionally, in $\mathbb{R}^3$ the **cross product** (vector product) of two vectors $\vec{x}$ and $\vec{y}$ can be defined by

$$\vec{x} \times \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} = -\vec{y} \times \vec{x}\,. \tag{1.8}$$

This product is related to the included angle ϕ, too:

$$|\vec{x} \times \vec{y}| = |\vec{x}|\,|\vec{y}|\,|\sin(\phi)| \tag{1.9}$$

From this relation the following properties of the vector product can be obtained:

(i) $\vec{x} \times \vec{y}$ is perpendicular to $\vec{x}$ and $\vec{y}$.

(ii) $|\vec{x} \times \vec{y}|$ can be interpreted as the unsigned area of the parallelogram having $\vec{x}$ and $\vec{y}$ as sides.

(iii) $\{\vec{x}, \vec{y}, \vec{x} \times \vec{y}\}$ is right-handed.

(iv) $\vec{x} \times \vec{y} = 0$ if $\vec{x}||\vec{y}$.

A double cross product can be related to the dot product by the triple product expansion:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b}) \tag{1.10}$$

1.3 Coordinate Systems

In order to uniquely identify a position of an object a vector has to be combined with a point of origin. Besides the **Cartesian coordinate system** used in Eq. (1.5), the **polar coordinate system** has proven useful for systems with rotational symmetries (e.g. a circular motion). This coordinate system include a radial and an angular coordinate (r, ϕ), which can be connected to suitable Cartesian coordinates (x_1, x_2):

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} \quad \text{with} \quad r = \sqrt{x_1^2 + x_2^2}, \qquad \tan\phi = \frac{x_2}{x_1} \tag{1.11}$$

Chapter 2

Basic Concepts

2.1 Physics and the Scientific Method

The word physics stems form the Greek word $\phi\upsilon\sigma\iota\kappa\eta$, which simple means nature. The aim of physics is to model the inanimate part of our environment based on reproducible observations. From this model understanding about the underlying laws of nature is obtained as well as predictions are deduced which can then be used to test the model (see Fig. 2.1). The model

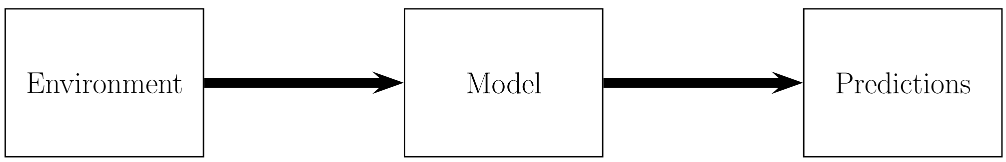

Figure 2.1: Modeling in physics: Events from the inanimate nature are modeled, in order to obtain an deeper understanding and make further predictions about nature.

is usually expressed in the form of only a few **axioms** or postulates, while the **deduction** of the predictions are usually performed by the help of mathematics. However, while some predictions of such a theory will be verified, some might be contradicted by **experiment**. These experiments can then be used to formulate **improved models**. Since the old theory still has proven within a certain "range", the improved model can still be tested against the old one within this range.
For instance classical mechanics is fully determined by the three Newton's laws. However, it fails to describe physical processes on the atomic level and

has to be extended to quantum mechanics there. Additionally, if one deals with huge velocities (compared to the velocity of light) classical mechanics has to be extended to the special theory of relativity.

2.2 Physical Quantities

Measuring the predicted quantities of a model means comparing these quantities to a common standard. This standard is expressed by the unit when stating the value of this quantity. For instance the angle, introduced in Eq. (1.7), can be compared to the seconds on a clock face or to the length of the circumference of a unit circle. The former one is denoted in degree with symbol ° and the latter one in radian with symbol rad. According to custom, only in the latter case or when the quantity vanishes the unit is omitted.
In this lecture **The International System of Units (SI)** will be used, which has the following base units:

Physical quantity	Unit	Symbol
Time	Second	s
Length	Meter	m
Mass	Kilogram	kg
Electric Current	Ampere	A
Temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

Besides these base units, combined ones like the Newton $\mathrm{N} = \mathrm{kg\,m^2/s}$ are also defined. Additionally, the units can be accompanied by a prefix, e.g. one of the following:

Factor	10^9	10^6	10^3	10^{-2}	10^{-3}	10^{-6}	10^{-9}	10^{-12}
Prefix	giga	mega	kilo	centi	milli	micro	nano	pico
Shortcut	G	M	k	c	m	μ	n	p

The order of magnitude, which is described by these prefixes, can also be written explicitly when using **scientific notation**. In this notation leading zero-digits do not appear, therefore the first digit is followed by the decimal point. Thus it states most clearly the **significant digits** of a physical quantity. The statement of a (possible) measurement of a physical quantity is therefore expressed in the following pattern:

$$\text{(physical value)} = \text{(numeric value)}\ [\times\ 10^{\text{order of magnitude}}]\ \text{[prefix](unit)} \tag{2.1}$$

Part I

Classical Mechanics

Chapter 3

Kinematics and Kinetics

3.1 Motions

As stated in section 1.3 the position of a particle is described by a vector $\vec{r}$ for a given coordinate system with fixed point of origin. Therefore the motion is characterized by a function of this quantity in time t. Mathematically this is a mapping $[t_1, t_2] \subset \mathbb{R} \rightarrow \mathbb{R}^d$ where $d = 1, 2, 3$ is the dimension of the problem. For simplicity we will start with a **linear motion** $d = 1$, that means we are looking at the position

$$x(t), \qquad [x(t)] = \mathrm{m}\,. \tag{3.1}$$

The **velocity** or speed $v(t)$ of a particle specifies the (linear) change of this quantity and therefore is given by

$$v(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \dot{x}(t) \qquad [v(t)] = \frac{\mathrm{m}}{\mathrm{s}}\,, \tag{3.2}$$

where the derivative with respect to the time t is denoted with a dot, which is the common notation in physics. In analogy the **acceleration** $a(t)$ of a particle characterizes the change of the velocity:

$$a(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} = \dot{v}(t), \qquad [a(t)] = \frac{\mathrm{m}}{\mathrm{s}^2}\,. \tag{3.3}$$

As we will see later on, constant acceleration $a(t) = a$ is quite common in physics. In this case, integrating Eq. (3.3) once the velocity is obtained:

$$v(t) = \int a \,\mathrm{d}t = at + v_0 \tag{3.4}$$

with an integration constant v_0. Therefore the velocity increases ($a > 0$) or decreases ($a < 0$) linearly with time at constant acceleration. In the case

$a = 0$ the velocity stays constant. This is called a **uniform motion**.
To obtain $x(t)$ we integrate Eq. (3.4) once more and obtain

$$\boxed{x(t) = \frac{a}{2}t^2 + v_0 t + x_0}. \tag{3.5}$$

This is the path of the particle in one dimension under a constant acceleration. There are two undetermined integration constants v_0 and x_0. Thus the motion of the particle is uniquely defined if the position and the velocity at the same or different times *or* two positions at different times *or* two velocities at different times are specified.

EXAMPLE A particle is accelerated with $a = 10\,\frac{\text{m}}{\text{s}^2}$. At time $t = 0\,\text{s}$ the particle is at $x(0\,\text{s}) = 0\,\text{m}$ and at time $t = 2\,\text{s}$ at $x(2\,\text{s}) = 40\,\text{m}$. Determine the position $x(t)$ of the particle with respect to time.
We have

$$\begin{aligned} 0\,\text{m} &= x(0\,\text{s}) = x_0 \\ 40\,\text{m} &= x(2\,\text{s}) = 20\,\text{m} + 2\,\text{s} \cdot v_0 \end{aligned}$$

Thus $x_0 = 0\,\text{m}$ and $v_0 = 10\,\frac{\text{m}}{\text{s}}$ and we get:

$$\underline{\underline{x(t) = 5\,\frac{\text{m}}{\text{s}^2} \cdot t^2 + 10\,\frac{\text{m}}{\text{s}} \cdot t}}$$

By the use of the description of the components of a vector Eq. (1.4), these definitions and calculations can easily be generalized to $d = 3$ dimensions. In this case and again at constant acceleration $\vec{a}$ one has in vector or component notation

$$\vec{r}(t) = \frac{\vec{a}}{2}t^2 + \vec{v}_0 t + \vec{r}_0 \qquad \text{or} \qquad \left\{ \begin{array}{lcl} x(t) &=& \frac{a_x}{2}t^2 + v_x t + x_0 \\ y(t) &=& \frac{a_y}{2}t^2 + v_y t + y_0 \\ z(t) &=& \frac{a_z}{2}t^2 + v_z t + z_0 \end{array} \right. . \tag{3.6}$$

EXAMPLE Assume a particle that is at time $t = 0$ at $(0, 0, 0)$ and has only velocity $v_x = 1\,\frac{\text{m}}{\text{s}}$ in the x-direction, and which is accelerated through gravity by $-g = -10\,\frac{\text{m}}{\text{s}^2}$ in the y-direction. State $\vec{r}(t)$ and sketch the motion.
For $\vec{r}(t)$ we easily find

$$\underline{\underline{\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} v_x t \\ -\frac{g}{2}t^2 \\ 0 \end{pmatrix}}}$$

If we solve the equation for $x(t)$ with respect to t and insert this expression for the time in the equation for $y(t)$, we obtain a parabola

$$y(x) = -\frac{g}{2}\frac{x^2}{v_x^2} = -5\,\mathrm{m}^{-1} \cdot x^2\,.$$

The last exercise serves as a good example for the so called principle of superposition. We are dealing here with two independent motions: On the one hand we have the motion in x-direction with a constant velocity (i.e. no acceleration) and on the other hand we have have motion in the z-direction with a linearly increasing velocity (i.e. constant acceleration). These two motions are simply added

$$\vec{r}(t) = \begin{pmatrix} v_x t \\ 0 \\ -\frac{g}{2}t^2 \end{pmatrix} = \begin{pmatrix} v_x t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{g}{2}t^2 \end{pmatrix}. \tag{3.7}$$

This is called the **principle of superposition**.

3.2 Circular Motion 1

Another important kind of motion is the **circular motion** as depicted in Fig. 3.1. We consider a particle moving on a circle in the x-y-plane with radius r. For this kind of motion it is more convenient to deal with the **angular velocity** $\omega(t)$:

$$\omega(t) = \dot{\phi}(t), \quad [\omega] = \frac{1}{\mathrm{s}} \tag{3.8}$$

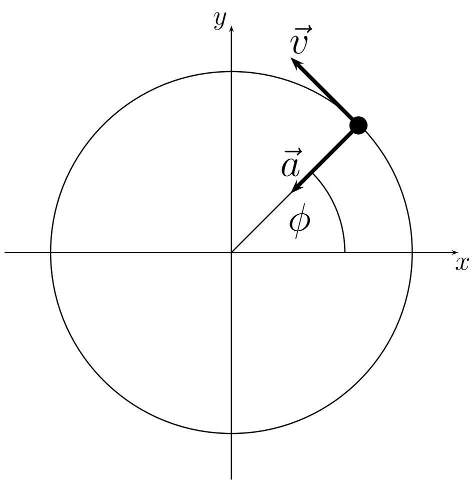

Figure 3.1: Circular motion of a particle.

We are interested in a constant angular velocity, that is $\ddot{\phi}(t) = \dot{\omega}(t) = 0$ and therefore

$$\omega(t) = \omega = \frac{\Delta\phi}{\Delta t} = \frac{2\pi}{T} \tag{3.9}$$

where T is the **time of circulation** or period. Often the (physical) **frequency** f of the circulation defined by

$$f = \frac{\omega}{2\pi} = \frac{1}{T}, \qquad [f] = \frac{1}{\mathrm{s}} \tag{3.10}$$

is used to characterize the time of circulation. For $\phi(t)$ we get:

$$\phi(t) = \int \omega \, \mathrm{d}t = \omega t + \phi_0 \tag{3.11}$$

If we introduce polar coordinates, set $\phi_0 = 0$ we are able to specify the position of the particle in Cartesian coordinates

$$\boxed{\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \end{pmatrix}}. \tag{3.12}$$

For the velocity $\vec{v}(t)$ we obtain

$$\vec{v}(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \omega \begin{pmatrix} -r\sin(\omega t) \\ r\cos(\omega t) \end{pmatrix} \tag{3.13}$$

It can be easily seen, that for each time t it holds $\vec{r}(t) \perp \vec{v}(t)$ since

$$\vec{r}(t) \cdot \vec{v}(t) = \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \end{pmatrix} \cdot \begin{pmatrix} -r\sin(\omega t) \\ r\cos(\omega t) \end{pmatrix} \tag{3.14}$$

$$= r^2\omega\big(-\cos(\omega t)\sin(\omega t) + \sin(\omega t)\cos(\omega t)\big) = 0 \tag{3.15}$$

If we are interested in the absolute value of the velocity, we have to take the the absolute value of $\vec{v}(t)$ in Eq. (3.13) resulting in

$$|\vec{v}(t)| = v = \omega r\,. \tag{3.16}$$

Another way to obtained this result is simply noting that the particle needs $T = \frac{2\pi}{\omega}$ to go one time around the circle and covering thereby a distance of $s = 2\pi r$, such that

$$v = \frac{2\pi r}{\frac{2\pi}{\omega}} = \omega r\,. \tag{3.17}$$

Since $\vec{r}(t) \perp \vec{v}(t)$ we can write

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}} \tag{3.18}$$

where $\vec{\omega}$ points outside the x-y plane along the axis of rotation. For the acceleration $\vec{a}$ we finally obtain:

$$\vec{a}(t) = \begin{pmatrix} a_x(t) \\ a_y(t) \end{pmatrix} = \begin{pmatrix} \dot{v}_x(t) \\ \dot{v}_y(t) \end{pmatrix} = -\omega^2 \begin{pmatrix} r\cos(\omega\, t) \\ r\sin(\omega t) \end{pmatrix} = -\omega^2 \vec{r}(t) \tag{3.19}$$

We again see, that $\vec{a}(t) \perp \vec{v}(t)$, and that the acceleration is always pointing in the negative $\vec{r}(t)$ direction. The absolute value $a_C = |\vec{a}(t)|$ is given by

$$\boxed{a_C = \omega^2 r} \tag{3.20}$$

This is called **centripetal acceleration**.

3.3 Summary: Kinematics

$\vec{a}$	Kind of motion	Mathematical description
$\vec{a} = \vec{0}$	uniform	$x(t) = v_0 t + x_0$
$\vec{v} \parallel \vec{a}$, $\vec{a} = const$	linear motion with constant acceleration	$x(t) = \frac{1}{2}at^2 + v_0 t + x_0$
$\vec{v} \not\parallel \vec{a}$, $\vec{a} = const$	planar motion: 2-dimensional superposition of a uniform one (e.g. in x-direction and a motion with constant acceleration (e.g. in y-direction)	Maximal height $h = \frac{{v_{y,0}}^2}{2a}$ reached at $\tilde{T} = \frac{v_{y,0}}{a}$ Same initial height at $T = 2\tilde{T}$ defining the range $s = \frac{v_{x,0} v_{y,0}}{a}$
$\vec{v} \perp \vec{a}$, $a = const$	circular	$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r \begin{pmatrix} \cos(\omega t + \phi) + x_0 \\ \sin(\omega t + \phi) + y_0 \end{pmatrix}$, with $v = \omega r$ and $a = \omega^2 r$
$\vec{v} \not\perp \vec{a}$, $a = const$	helical	$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} r\cos(\omega t + \phi) + x_0 \\ r\sin(\omega t + \phi) + y_0 \\ \frac{1}{2}a_z t^2 + v_{z,0} t + z_0 \end{pmatrix}$, with $\sqrt{{v_{x,0}}^2 + {v_{y,0}}^2} = \omega r$ and $\sqrt{{a_{x,0}}^2 + {a_{y,0}}^2} = \omega^2 r$

3.4 Newton's Axioms

So far we dealt only with the question how to describe the motion of a mass point (that is the path $\vec{r}(t)$ of the mass point). This section addresses the question, what makes a body moving and changing its state of motion. The answer is: A **force** exactly does this. Isaac Newton (1643-1727) was the first who realize that for a given body the absolute value of the force is proportional to the absolute value of the acceleration. The constant of proportionality is called the (inertial) **mass** m. This is **Newton's second axiom** (the principle of action):

$$\boxed{\vec{F} = m\ddot{\vec{r}}} \qquad [m] = \mathrm{kg}, \qquad [F] = \mathrm{N} = \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2} \tag{3.21}$$

In defining the **momentum** as $\vec{p} = m\dot{\vec{r}} = m\vec{v}$, the axiom simplifies to

$$\vec{F} = \dot{\vec{p}}, \qquad [\vec{p}] = \mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}}\,. \tag{3.22}$$

Even though the last two equations define the mass m, for practical purposes it is more important to ask the following two questions:

(a) What is $\ddot{\vec{r}}$, if $\vec{F}$ and m are given?

(b) What is $\vec{F}$, if $\ddot{\vec{r}}$ and m are given?

We will mainly address the first question during this lecture. If the force $\vec{F}$ in Eq. (3.21) is zero, we see that $\vec{v}$ is constant. This is **Newton's first axiom** (principle of inertia):
Each body, on which no force acts stays at rest or moves with a constant velocity.
In short notation this axiom may be written as

$$\boxed{\sum_i \vec{F}_i = 0 \qquad \Leftrightarrow \qquad \vec{v} = \mathrm{const}}\,. \tag{3.23}$$

In addition **Newton's third axiom** (actio = reactio) states, that if a body 1 acts on a body 2 by a force $\vec{F}_{12}$, then body 2 acts on body 1 with a force $\vec{F}_{21}$, such that

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}\,. \tag{3.24}$$

3.5 Forces

In this section the most common forces are introduced: Foremost, it should be emphasized that for all forces as well as for velocities holds the **principle of superposition**, i.e. forces may be composed and decomposed.

Gravity
The gravitational force between two bodies with (gravitational) masses m_1, m_2 which are separated by a distance r is given by

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{\vec{r}} \qquad \text{or} \qquad \boxed{F = G\frac{m_1 m_2}{r^2}} \tag{3.25}$$

with the unit vector $\hat{\vec{r}} = \frac{\vec{r}}{r}$ and the gravitational constant

$$G = 6,67 \cdot 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2} \tag{3.26}$$

If we are interested in the absolute value of the gravitational force near the earth we set $m_2 = m_e = 5,97 \cdot 10^{24}\,\mathrm{kg}$ and $r = r_e = 6370\,\mathrm{km}$ and get the simplified version

$$\boxed{F_G = m\frac{G\,m_e}{r_e^2} \equiv mg} \tag{3.27}$$

with Earth's gravity $g = 9,81\,\frac{\mathrm{m}}{\mathrm{s}^2}$ (in the exercises we will often use $g = 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$).

EXAMPLE A common example is the calculation of the acceleration of a mass m on an **inclined plane**. As shown in Fig 3.2 we decompose the gravitational force F_G in the two components normal force F_N and downhill-slope force F_D which are given by

$$\begin{aligned} F_D &= F_G \sin\alpha \\ F_N &= F_G \cos\alpha \end{aligned}$$

Therefore the acceleration of the mass is given by Newton's second axiom:

$$\underline{a} = \frac{F_D}{m} = \frac{mg}{m}\sin\alpha = \underline{\underline{g\sin\alpha}}$$

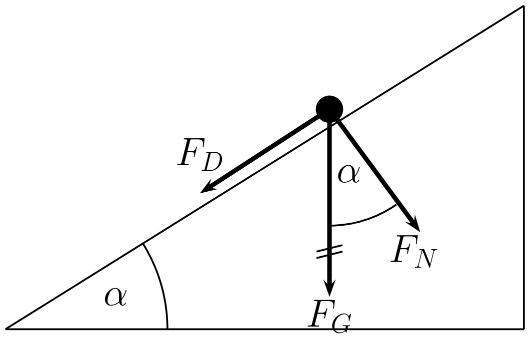

Figure 3.2: Decomposition of forces on an inclined plane.

Hook's law
The force of a spring is given by (in $d = 1$):

$$\boxed{F_H = -Dx} \qquad [D] = \frac{\text{N}}{\text{m}} \tag{3.28}$$

with the spring constant D.

Friction
We will consider two kinds of friction forces:

(i) Static friction: A body which is attached to another one will be prevented to start moving. The corresponding maximal force is given by

$$F_S = \mu\, F_N \tag{3.29}$$

with the coefficient of static friction μ with values usually between 0.02 and 1.

EXAMPLE As an example we consider again a body with mass m on an inclined plane and ask at which angle it will begin to move. Again, the following forces are relevant:

$$\begin{aligned} F_D &= mg \sin\alpha \\ F_N &= mg \cos\alpha \end{aligned}$$

The body just starts to move if $F_D = \mu F_N$, that is

$$m\, g \sin\alpha = \mu\, m\, g \cos\alpha \qquad \text{or} \qquad \underline{\underline{\tan\alpha = \mu}}\ .$$

(ii) Stokes friction: If an object is moving through a media (e.g. air), the friction force is given by

$$F_R = -\gamma_S \, v \tag{3.30}$$

with the Stokes friction coefficient γ_S. This friction will play a role when we discuss damped oscillations.

Centripetal and Centrifugal Force
In section 3.2 we saw, that an object moving on a circle with some constant velocity v has a changing direction of motion, i.e. the vector $v(t)$ changes its direction. The rate of this change in velocity is the centripetal acceleration $a_C = \omega^2 r = \frac{v^2}{r}$ with $v = \omega r$, caused by the Centripetal force

$$F_C = \frac{mv^2}{r} \tag{3.31}$$

The corresponding fictitious force which would be felt by another object inside the moving one is called centrifugal force.

3.6 Work and Conservation Laws

Generally **work** is defined by

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r}, \quad [W] = \mathrm{J} \tag{3.32}$$

We again constrict ourself to the one dimensional case where this formula reduces to

$$\boxed{W = \int_{x_1}^{x_2} F(x) \, \mathrm{d}x}. \tag{3.33}$$

EXAMPLES

(i) Work needed to uplift a body with mass m under the influence of gravitation from 0 to h:

$$\underline{\underline{E_{\text{pot}}}} = \int_0^h F_G \, \mathrm{d}x = \int_0^h mg \, \mathrm{d}x = \underline{\underline{mgh}} \tag{3.34}$$

(ii) Work needed to move a body with mass m attached to a spring with spring constant D from position 0 to s:

$$\underline{\underline{E_{\text{pot}}}} = \int_0^s F_H \, \mathrm{d}x = \int_0^s Dx \, \mathrm{d}x = \underline{\underline{\frac{1}{2} D s^2}} \tag{3.35}$$

(iii) Work needed needed to accelerate a particle with mass m from 0 to v_0 (kinetic energy):

$$\underline{\underline{E_{\text{kin}}}} = \int F\,\mathrm{d}x = \int m\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}x = m\int_0^{v_0} v\,\mathrm{d}v = \underline{\underline{\frac{m}{2}v_0^2}} \tag{3.36}$$

Now we will discuss **energy** and the **conservation of energy**. We consider a particle with mass m subject to a force $F = F(x)$ in one dimension, i.e. its motion is governed by the second Newton's law $F\big(x(t)\big) = m\ddot{x}(t)$. Let us define

$$V(x) = -\int F(x)\,\mathrm{d}x \qquad \Leftrightarrow \qquad F(x) = -V'(x) \tag{3.37}$$

where $V(x)$ is called **potential**. With this definition Newtons second axiom reads:

$$-V'\big(x(t)\big) = m\ddot{x}(t) \tag{3.38}$$

In multiplying this equation by $\dot{x}(t)$ and using the chain rule a conservation law is obtained:

$$-V'\big(x(t)\big)\dot{x}(t) = m\ddot{x}(t)\dot{x}(t) \tag{3.39}$$

$$\Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{m}{2}\dot{x}^2 + V(x)\Big) = 0 \tag{3.40}$$

Thus, the expression in the brackets is constant (as a function of time) and called **energy** E. Using $v = \dot{x}$ the conservation of energy reads

$$\boxed{E = \frac{m}{2}v^2 + V(x)} \tag{3.41}$$

meaning the sum of kinetic and potential energy remains constant.

EXAMPLES

(i) Energy of a (moving) body with mass m subject to gravity:

$$F_G(x) = -mg \quad \Rightarrow \quad V(x) = mgx \quad \Rightarrow \quad E = \frac{m}{2}v^2 + mgx \tag{3.42}$$

(ii) Energy of a (moving) mass m attached to a spring with constant D:

$$F_H(x) = -Dx \quad \Rightarrow \quad V(x) = \frac{D}{2}x^2 \quad \Rightarrow \quad E = \frac{m}{2}v^2 + \frac{D}{2}x^2 \tag{3.43}$$

Next we are going to consider the **momentum** of a system of *two* interacting particles. Newton's third law tells, that $\vec{F}_{12} = -\vec{F}_{21}$, that means

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2\Big) = 0 \tag{3.44}$$

Therefore, the **total momentum**

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 \tag{3.45}$$

is conserved.

EXAMPLE As an example we want to describe the motion of the center of two masses m_1 and m_2 which are only subjects their pairwise force. The center of the two masses is defined by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{1}{\mathcal{M}}(m_1\vec{r}_1 + m_2\vec{r}_2) \tag{3.46}$$

with the total mass $\mathcal{M} = m_1 + m_2$.
Its time derivative

$$\dot{\vec{R}} = \frac{1}{\mathcal{M}}(m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2) = \frac{1}{\mathcal{M}}(\vec{p}_1 + \vec{p}_2) = \frac{1}{\mathcal{M}}\vec{P} \tag{3.47}$$

is proportional to the total momentum $\vec{P}$. Since this quantity is conserved we can conclude that the center of the two masses performs an uniform motion .

This example can easily be generalized to a system with N particles (a two body interaction is still assumed). Since $\vec{F}_{ik} = -\vec{F}_{ki}$ we have $\sum_i \sum_{k\neq i} \vec{F}_{ik} = 0$ and due to the force acting on the particle i is $\vec{F}_i = \sum_{k\neq i} \vec{F}_{ki} = \dot{\vec{p}}_i$ we have

$$\dot{\vec{P}} = \sum_i \dot{\vec{p}}_i = \sum_i \sum_{k\neq i} \vec{F}_{ik} = 0 \tag{3.48}$$

and therefore we find again that the total momentum $\vec{P}$ is a constant of motion. Likewise, the **center of mass**

$$\vec{R} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} = \frac{1}{\mathcal{M}}\sum_i m_i\vec{r}_i \tag{3.49}$$

performs an uniform motion.

3.7 Applications: Collisions

In order to discuss an application we will consider the collision of two particles in one dimension, i.e. a process

$$v_1, v_2 \rightarrow v_1', v_2' \,. \tag{3.50}$$

We will take a closer look at two distinct situations, each with different conservation laws to be applied:

ELASTIC COLLISION	INELASTIC COLLISION
Conservation of Momentum $m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$	Conservation of Momentum $m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$
Conservation of energy $\frac{m_1}{2}{v_1}^2 + \frac{m_2}{2}{v_2}^2 = \frac{m_1}{2}{v_1'}^2 + \frac{m_1}{2}{v_2'}^2$	Energy is not conserved: Some energy is converted into heat $\frac{m_1}{2}{v_1}^2 + \frac{m_2}{2}{v_2}^2 = \frac{m_1}{2}{v_1'}^2 + \frac{m_1}{2}{v_2'}^2 + Q$

Taking into account the conservation of momentum and energy for the **elastic collision** we obtain:

$$m_1(v_1 - v_1') = m_2(v_2' - v_2) \tag{3.51}$$

$$m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2) \tag{3.52}$$

Using the third binomial law we get

$$m_1(v_1 - v_1') = m_2(v_2' - v_2) \tag{3.53}$$

$$m_1(v_1 - v_1')(v_1 + v_1') = m_2(v_2' - v_2)(v_2' + v_2) \,. \tag{3.54}$$

Thus we have

$$v_1 + v_1' = v_2 + v_2' \tag{3.55}$$

and together with the conservation of momentum

$$\underline{\underline{v_1' = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2}}} \quad \text{and} \quad \underline{\underline{v_2' = \frac{(m_2 - m_1)v_2 + 2m_1v_1}{m_1 + m_2}}} \,. \tag{3.56}$$

In the case of an **inelastic collision** we will only consider the special case of $v_2 = 0$ and that the two bodies stick together right after the collision, i.e. they are moving with the same velocity $v_1' = v_2'$. Such a collision is called **total inelastic collision**. Thus, the conservation laws

$$m_1v_1 = (m_1 + m_2)v_1' \tag{3.57}$$

$$\frac{m_1}{2}v_1^2 = \frac{m_1 + m_2}{2}{v_1'}^2 + Q \tag{3.58}$$

yield

$$\underline{\underline{v_1' = \frac{m_1}{m_1 + m_2}v_1}} \quad \text{and} \quad \underline{\underline{Q = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}v_1^2}} \,. \tag{3.59}$$

3.8 Circular Motion 2

We consider again a particle moving around some axis, which we will call $\vec{\omega}$. The velocity is given by

$$\vec{v}(t) = \vec{\omega} \times \vec{r}(t)\,. \tag{3.60}$$

If the vector $\vec{r}(t)$ is in a plane perpendicular to the axis ω this formula simplifies to $v = \omega r$ (see sec. 3.2), where $r = |\vec{r}|$ is the distance from the axis to the particle. Since for rotation the momentum of the particle is not conserved it is better to look at the **angular momentum** $\vec{L}$ or correspondingly the **torque** $\vec{M}$ which are defined by

$$\boxed{\vec{L} = \vec{r} \times \vec{p}} \qquad [\vec{L}] = \mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}} \tag{3.61}$$

$$\boxed{\vec{M} = \vec{r} \times \vec{F}} \qquad [\vec{M}] = \mathrm{kg}\frac{\mathrm{m}^2}{\mathrm{s}^2} \tag{3.62}$$

Since $\dot{\vec{r}}$ is parallel to $\vec{p} = m\dot{\vec{r}}$ and with $\dot{\vec{p}} = \vec{F}$ we obtain

$$\boxed{\dot{\vec{L}} = \vec{M}} \tag{3.63}$$

With respect to Eq. (3.60) the kinetic energy of this particle may be rewritten as the **rotational energy**

$$E_r = \frac{m}{2}\vec{v}^2 = \frac{m}{2}(\vec{\omega} \times \vec{r})^2 = \frac{m}{2}\omega^2 r^2 \tag{3.64}$$

In the last step we assumed again an axis perpendicular to the plane of motion. Defining the **moment of inertia** Θ by

$$\Theta = mr^2 \tag{3.65}$$

we can rewrite the energy as

$$\boxed{E_r = \frac{\Theta}{2}\omega^2} \tag{3.66}$$

3.9 Application: Planetary Motion

Devoted to the motions of planets in our solar system, the Kepler problem deals with the motion of two masses m_1 and m_2 under the influence of the gravitational force

$$\vec{F}_{12} = -G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) = -\vec{F}_{21}\,. \tag{3.67}$$

Kepler's laws contain the following statements:

(1) The orbit of a planet around a star is an ellipse with the star at one focus.

(2) A line joining a planet and its star sweeps out equal areas during equal intervals of time. This is known as the law of equal areas, too.

(3) The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits.

We will first prove the second statement. Since $\vec{F} \propto \vec{r}$, we see from Eq. (3.62) that $\vec{M} = 0$ and consequently $\vec{L}$ is a constant of motion:

$$\vec{L} = const \tag{3.68}$$

In particular that tells us, that the motion takes place in a plane. From Fig. 3.3 we can see that the area the position vector covers during an infinitesimal time is

$$\mathrm{d}A = \frac{1}{2}|\vec{r} \times \mathrm{d}\vec{r}|\,. \tag{3.69}$$

Taking the time derivative we obtain

$$\underline{\underline{\dot{A}}} = \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\left|\vec{r} \times \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right| = \frac{1}{2m}|\vec{r} \times \vec{p}| = \frac{L}{2m} = \underline{\underline{const}}\,. \tag{3.70}$$

This is the mathematical formulation of Kepler's second law.
For the first and third law let us first recall the definition of the center of mass $\vec{R} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)$ with $\mathcal{M} = m_1 + m_2$. If we furthermore introduce the **relative distance** $\vec{r} = \vec{r}_1 - \vec{r}_2$ between the two masses we can express the individual positions through this quantities by

$$\vec{r}_1 = \vec{R} + \frac{m_2}{\mathcal{M}}\vec{r} \tag{3.71}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{\mathcal{M}}\vec{r}\,. \tag{3.72}$$

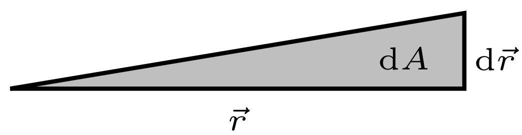

Figure 3.3: Derivation of Newton's second law. The solid filled triangular is the area $\mathrm{d}A = \frac{1}{2}|\vec{r} \times \mathrm{d}\vec{r}|$.

Since the center of mass $\vec{R}$ is a constant of motion we can choose our coordinate system such that the origin coincides with the center of mass. We obtain

$$\vec{r}_1 = +\frac{m_2}{\mathcal{M}}\vec{r} \tag{3.73}$$

$$\vec{r}_2 = -\frac{m_1}{\mathcal{M}}\vec{r}. \tag{3.74}$$

Thus, the equation of motion

$$m_1\ddot{\vec{r}}_1 = -G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \tag{3.75}$$

may be rewritten as

$$\frac{m_1 m_2}{\mathcal{M}}\ddot{\vec{r}} = -G\frac{m_1 m_2}{r^3}\vec{r} \tag{3.76}$$

with $r = |\vec{r}|$. Since the angular momentum $\vec{L}$ is a constant of motion the motion is situated in a plane, which we will choose to be the $x-y$ plane (by a suitable coordinate system). Furthermore we will only prove a special (and easier) case of an elliptic motion – the circular motion. We make the ansatz

$$\vec{r}(t) = \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \\ 0 \end{pmatrix} \quad \Rightarrow \quad \ddot{\vec{r}}(t) = -\omega^2 \begin{pmatrix} r\cos(\omega t) \\ r\sin(\omega t) \\ 0 \end{pmatrix} . \tag{3.77}$$

This yields

$$-\frac{1}{m_1 + m_2}\omega^2 = -G\frac{1}{r^3}\,, \tag{3.78}$$

and with $\omega = \frac{2\pi}{T}$ (T being the time of circulation) we finally obtain

$$\underline{\underline{T^2 = \frac{4\pi^2}{G(m_1 + m_2)}\,r^3}} \;. \tag{3.79}$$

That proves the first and third law.

3.10 Motion of a Rigid Body

A simple rigid body can be defined as N discrete masses $m_i,\ i = 1, \ldots, N$ with pairwise fixed distances. The positions of the individual masses will be denoted by $\vec{r}_i$. With these definitions the kinetic energy takes the form

$$E = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\vec{r}}_i)^2\,. \tag{3.80}$$

Using the coordinate of the center of mass $\vec{R} = \frac{1}{\mathcal{M}} \sum_i m_i \vec{r}_i$ and relative coordinates $\vec{r}_i{}' = \vec{r}_i - \vec{R}$, we can rewrite the energy as

$$E = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{R}}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{\vec{r}}_i{}')^2 + \dot{\vec{R}} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i{}' \,. \tag{3.81}$$

Since the definition of the new coordinates the latter sum will vanish:

$$\sum_{i=1}^{N} m_i \dot{\vec{r}}_i - \sum_{i=1}^{N} m_i \dot{\vec{R}} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i - \dot{\vec{R}} \mathcal{M} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i = 0 \tag{3.82}$$

and therefore the kinetic part of the energy results in

$$E = \frac{1}{2} \left(\sum_{i=1}^{N} m_i \right) \dot{\vec{R}}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{\vec{r}}_i{}')^2 \,. \tag{3.83}$$

The latter term can be rewritten when considering the body rotating around an axis $\vec{\omega}$ through the center of mass. Since the velocity is then given by $\dot{\vec{r}}_i{}' = \vec{\omega} \times \vec{r}_i{}'$ and therefore with $(\vec{\omega} \times \vec{r}_i{}')^2 = \omega^2 (r'_{\perp,i})^2$ the energy is given by

$$E = \frac{1}{2} \left(\sum_{i=1}^{N} m_i \right) \dot{\vec{R}}^2 + \frac{1}{2} \omega^2 \sum_{i=1}^{N} m_i (r'_{\perp,i})^2 \tag{3.84}$$

$$= \frac{1}{2} \mathcal{M} \dot{\vec{R}}^2 + \frac{1}{2} \Theta \omega^2 \,. \tag{3.85}$$

where we defined the **moment of inertia** for a system consisting of N particles

$$\Theta = \sum_{i=1}^{N} m_i (r'_{\perp,i})^2 \,. \tag{3.86}$$

Since the energy of the rigid body decompose in two parts $E = E_{\text{kin}} + E_{\text{rot}}$ we conclude that the motion of the rigid body consists always of a translation and a rotation which are independent. The part of the translation contains the movement of the center of mass with total mass $\mathcal{M}$ and the part for the rotational energy is given by the rotation of a body with moment of inertia Θ with angular velocity ω.

We now turn to the **angular momentum** of rigid body. We consider again a rotation about an axis $\vec{\omega}$ through the center of mass $\vec{R}$. Furthermore we assume the axis $\vec{\omega}$ to be a symmetry axis of the body. The angular momentum is given by

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \,. \tag{3.87}$$

In decomposing the vector $\vec{r}$ in a component $\vec{r}_{||}$ parallel to $\vec{\omega}$ and a component $\vec{r}_\perp$ perpendicular to $\vec{\omega}$, we obtain for the summand

$$\begin{aligned} \vec{r} \times (\vec{\omega} \times \vec{r}) &= (\vec{r}_\perp + \vec{r}_{||}) \times \big(\vec{\omega} \times (\vec{r}_\perp + \vec{r}_{||})\big) = (\vec{r}_\perp + \vec{r}_{||}) \times (\vec{\omega} \times \vec{r}_\perp) \\ &= \vec{r}_\perp \times (\vec{\omega} \times \vec{r}_\perp) + \vec{r}_{||} \times (\vec{\omega} \times \vec{r}_\perp) \\ &= \vec{\omega}(r_\perp)^2 - \vec{r}_\perp (r_{||}\omega) \,. \end{aligned} \tag{3.88}$$

Since we considering the sum of Eq. (3.87), the second term will vanish due to the body rotate around a symmetry axis. Therefore we can conclude

$$\boxed{\vec{L} = \Theta \vec{\omega}} \,. \tag{3.89}$$

with the moment of inertia

$$\Theta = \sum_i m_i (r_{\perp,i})^2 \,. \tag{3.90}$$

For one particle the formula for the angular momentum coincides with the one previously obtained Eq. (3.63). Like the derivation there, the equation of motion for the rotation of a rigid body is given by

$$\boxed{\dot{\vec{L}} = \vec{M}} \,. \tag{3.91}$$

In real applications we no longer deal with discrete mass points rather than with a **continuous body**. Thus, we make the displacement $m_i \to \mathrm{d}m_i$ and $\sum \to \int$, such that

$$\Theta = \int_V (r_\perp)^2 \, \mathrm{d}m \,. \tag{3.92}$$

where the Integral covers the volume V of the considered body. Introducing the **density** $\rho = \frac{\mathrm{d}m}{\mathrm{d}V}$ this may be rewritten for a constant density as

$$\boxed{\Theta = \rho \int_V (r_\perp)^2 \, \mathrm{d}V} \,. \tag{3.93}$$

EXAMPLE As an example we will calculate the moment of inertia of a cylinder with radius R, total mass m and height h The density thus reads $\rho = \frac{m}{\pi R^2 h}$. Using polar coordinates and $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\phi$ we obtain

$$\underline{\underline{\Theta}} = \frac{m}{\pi R^2 h}\int_0^h \mathrm{d}z \int_0^R r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi r^2 = \frac{m}{\pi R^2 h} h \frac{R^4}{4} 2\pi = \underline{\underline{\frac{1}{2} m R^2}}$$

Furthermore in applications one is often interested in rotations around an axis which does not go through the center of mass, e.g. when a rigid body is put at a fixed axis and therefore the pivot is known to be motionless. By denoting the distance of the axis and a parallel axis through the center of motion by d, meaning the coordinate regarding the new axis is given by $\vec{r}_i` = \vec{r}_i + \vec{d}$, we can determine the new moment of inertia:

$$\Theta = \sum_i m_i (\vec{r}_{\perp,i}`)^2 = \sum_i m_i (\vec{r}_{\perp,i})^2 + d^2 \sum_i m_i + 2\vec{d} \sum_i m_i \vec{r}_{\perp,i} \tag{3.94}$$

Since for relative coordinates regarding the center of mass $\sum_i m_i \vec{r}_i = 0$ holds, the last term as a projection of this relation vanishes and we obtain the **parallel-axis theorem** (Steiner's theorem):

$$\boxed{\Theta = \Theta_{\mathrm{CM}} + \mathcal{M} d^2} \tag{3.95}$$

where Θ_{CM} is the moment of inertia with respect to the parallel axis through the center of motion.

EXAMPLE As an example we derive the equation of motion of a cylinder with radius R and total mass m on an inclined plane and its acceleration as shown in Fig. 3.4.
The torque with respect to the touch point T is given by $|\vec{M}| = RF_D$, i.e. $|\vec{M}| = mgR\sin\alpha$. The angular momentum regarding this point is found by the parallel-axis to $L = \Theta\omega = (\Theta_{\mathrm{CM}} + mR^2)\omega$. Thus we obtain the equation of motion

$$\underline{\underline{mgR\sin\alpha}} = \underline{\underline{(\Theta_{\mathrm{CM}} + mR^2)\dot{\omega}}} \quad \Leftrightarrow \quad \dot{\omega} = \frac{mgR\sin\alpha}{\Theta_{\mathrm{CM}} + mR^2}$$

The center of mass CM is subsequently subject to an acceleration

$$\underline{\underline{a}} = R\dot{\omega} = \underline{\underline{\frac{mR^2}{\Theta_{\mathrm{CM}} + mR^2} g\sin\alpha}}$$

Note that since $\Theta_{\mathrm{CM}} = \frac{1}{2} mR^2$ this acceleration $a = \frac{2}{3} g \sin\alpha$ is smaller than the one of a point mass $a_{PM} = g\sin\alpha$.

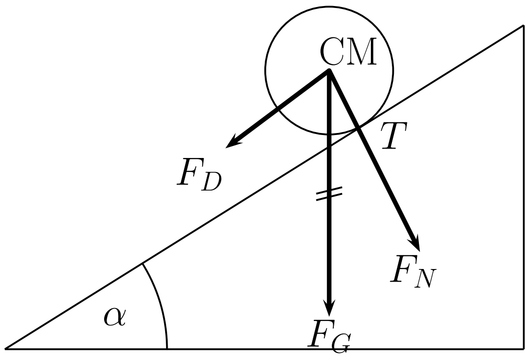

Figure 3.4: A cylinder on an inclined plane.

3.11 Summary: Translation and Rotation

At the end of this chapter we conclude with some correspondences between translations and rotations:

Translation	$\longleftrightarrow$	Rotation
Position $\vec{r}$	$s = \phi r$	Angle ϕ
Velocity $\vec{v} = \dot{r}$	$v = \omega r$	Angular velocity $\omega = \dot{\phi}$
Force $\vec{F}$	$\vec{M} = \vec{r} \times \vec{F}$	Torque $\vec{M}$
$\vec{F} = \dot{\vec{p}}$	Equation of motion	$\vec{M} = \dot{\vec{L}}$
Momentum $\vec{p} = m\vec{v}$	$\vec{L} = \vec{r} \times \vec{p}$	Angular momentum $\vec{L} = \Theta\vec{\omega}$
Mass m	$\Theta = \sum_k m_k {r_{\perp,k}}^2 = \int \rho {r_\perp}^2 \,\mathrm{d}V$	Moment of inertia Θ
Total mass $\mathcal{M} = \sum_k m_k$	$\Theta_O = \Theta_{CM} + \mathcal{M}d^2$	Total moment of inertia $\Theta_O = \sum_k \Theta_{O,k}$
Total Momentum $\vec{P} = \sum_k \vec{p}_k$		Total angular momentum $\vec{L}_O = \sum_k \vec{L}_{O,k}$
Kinetic energy $E_{\text{kin}} = \frac{m}{2}v^2$		Rotational energy $E_r = \frac{\Theta}{2}\omega^2$

Chapter 4

Oscillations and Waves

4.1 Simple Harmonic Oscillator

We consider again a body with mass m in one dimension which is attached to a spring. Therefore the mass is subject to a force $F = -Dx$. Applying Newton's principle of action ($F = m\ddot{x}$) we get a (second order and linear) differential equation for $x(t)$

$$\boxed{m\ddot{x}(t) = -Dx(t)}\,. \tag{4.1}$$

Its general solution, as shown in Fig. 4.1a, reads

$$\boxed{x(t) = A\cos(\omega\, t + \phi)}\,, \tag{4.2}$$

with the **amplitude** A and the **phase** ϕ to be determined by boundary conditions (for instance the position and velocity at time $t = 0$). To prove the solution we put Eq. (4.2) in Eq. (4.1) and get the condition

$$(m\omega^2 - D) \cdot A\cos(\omega\, t + \phi) = 0 \tag{4.3}$$

Therefore the (angular) frequency ω is determined by the spring constant D and the mass:

$$\boxed{\omega = \sqrt{\frac{D}{m}}} \qquad \Leftrightarrow \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{D}} \tag{4.4}$$

This kind of motion is called a **harmonic oscillation**. It occurs generally, whenever a mass point is subject to a force, whose magnitude is proportional to the amplitude and points in the opposite direction.
An even larger application range can be obtained when looking at a potential $V(x)$: Whenever a mass point is in a stable position of equilibrium x_0,

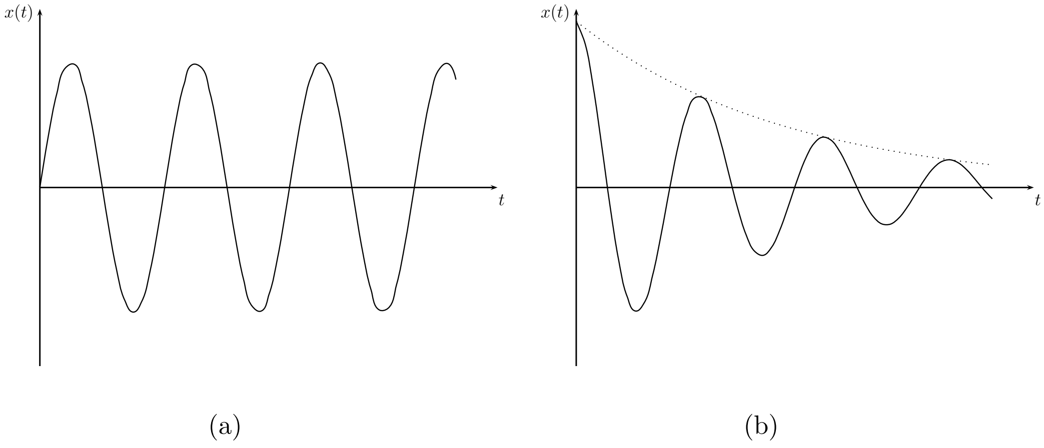

Figure 4.1: (a) Harmonic motion appears when a mass point is subject to a force $F = -Dx$. (b) Damped oscillation: The amplitude decays like $A\mathrm{e}^{-\lambda t}$.

meaning $V'(x_0) = 0$ and $V''(x_0) > 0$, the potential $V(x)$ for small deviations around x_0 may be approximated by a quadratic polynomial $V(x) = \frac{1}{2}Dx^2$ which then corresponds to a linear force. Thus, for small deviations many physical problems can be reduced to a harmonic oscillation.)

4.2 Applications: The Mathematical and the Physical Pendulum

A **mathematical pendulum** consists of a mass point m attached to a cord (assumed to be massless) with length ℓ subject to the gravitational force $F_G = -mg$ (see Fig.4.2). We may start from the expression $\dot{\vec{L}} = |\vec{M}|$ with $L = \Theta\omega = m\ell^2\dot{\alpha}$ and $|\vec{M}| = -mg\ell\sin\alpha$. We obtain the following differential equation

$$\ddot{\alpha}(t) = -\frac{g}{\ell}\sin\big(\alpha(t)\big)\,. \tag{4.5}$$

For small angles α we may approximate $\sin(\alpha) \approx \alpha$. Consequently the equation of motion reads

$$\ddot{\alpha}(t) = -\frac{g}{\ell}\alpha(t) \tag{4.6}$$

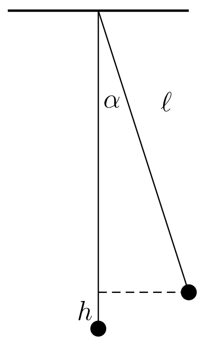

Figure 4.2: The Mathematical pendulum: A mass pony m is attached to a massless cord with length ℓ.

and we find the solution to be a harmonic oscillation

$$\alpha(t) = \alpha_0 \cos(\omega t) \tag{4.7}$$

with the angular frequency

$$\underline{\underline{\omega = \sqrt{\frac{g}{\ell}}}} \ . \tag{4.8}$$

In contrast the **physical pendulum** consists of an (arbitrary) body that can rotate about an axis A at distance ℓ from the center of mass S (see Fig. 4.3). The body has mass m and a moment of inertia Θ. We start from the equation of motion for a rigid body $|\vec{M}| = \dot{L} = \Theta\dot{\omega} = \Theta\ddot{\phi}$. The torque is given by $|\vec{M}| = -mg\ell \sin\phi$, such that

$$-mg\ell \sin\phi = \Theta\ddot{\phi} \, . \tag{4.9}$$

Again, for small angles $\sin\phi \approx \phi$ and we again obtain the common form of harmonic oscillations

$$\ddot{\phi} + \frac{mg\ell}{\Theta}\phi = 0 \, . \tag{4.10}$$

The solution reads $\phi(t) = \phi_o \cos(\omega t)$ with

$$\underline{\underline{\omega = \sqrt{\frac{mg\ell}{\Theta}}}} \ . \tag{4.11}$$

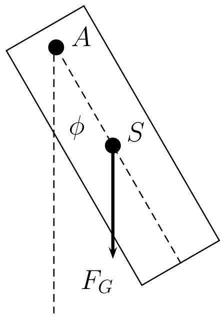

Figure 4.3: Physical pendulum with moment of inertia Θ with respect to the axis A. It is $\overline{AS} = \ell$ where S denotes the center of mass.

Comparing this with the formula $\omega = \sqrt{\frac{g}{\ell}}$ for the mathematical pendulum one may write

$$\omega = \sqrt{\frac{g}{\ell_r}} \qquad \text{with} \quad \ell_r = \frac{\Theta}{m\ell} \tag{4.12}$$

with the so called **reduced length of the pendulum** ℓ_r.

4.3 Damped Harmonic Oscillator

In a real system there will always be some kind of friction. A common case to consider would be stokes friction, which occurs for instance when the body is oscillating in a liquid. In addition to the reset force $-Dx$ we get a contribution $-\gamma\dot{x}$, such that the differential equation now reads

$$m\ddot{x}(t) + \gamma\dot{x}(t) + Dx(t) = 0\,. \tag{4.13}$$

With the abbreviations

$$\lambda = \frac{\gamma}{2m} \qquad \text{and} \quad \omega = \sqrt{\frac{D}{m} - \frac{\gamma^2}{4m^2}} \tag{4.14}$$

the solution is found to be

$$\boxed{x(t) = A\mathrm{e}^{-\lambda t}\cos(\omega t + \phi)}\,. \tag{4.15}$$

which is depicted in Fig. 4.1b. Therefore the amplitude of the oscillation decays exponentially with time when the object performs a **damped harmonic oscillation**

4.4 Enforced Oscillations

We consider now a simple harmonic oscillator (e.g. a mass m attached to a spring with spring constant D) subject to a harmonic (oscillating) external force $F_{\text{ext}} = F_0 \cos(\Omega t)$. Therefore the equation of motion takes the form

$$m\ddot{x}(t) = -Dx(t) + F_0 \cos(\Omega t) \tag{4.16}$$

$$\text{or} \quad \ddot{x}(t) + \omega^2 x(t) = \frac{F_0}{m} \cos(\Omega t)\,. \tag{4.17}$$

To solve this equation of motion we make the ansatz:

$$x(t) = A \cos(\Omega t) \tag{4.18}$$

Putting this ansatz in Eq. (4.17) we get

$$A(-\Omega^2 + \omega^2) = \frac{F_0}{m} \quad \Leftrightarrow \quad A = \frac{F_0}{m} \frac{1}{\omega^2 - \Omega^2}\,. \tag{4.19}$$

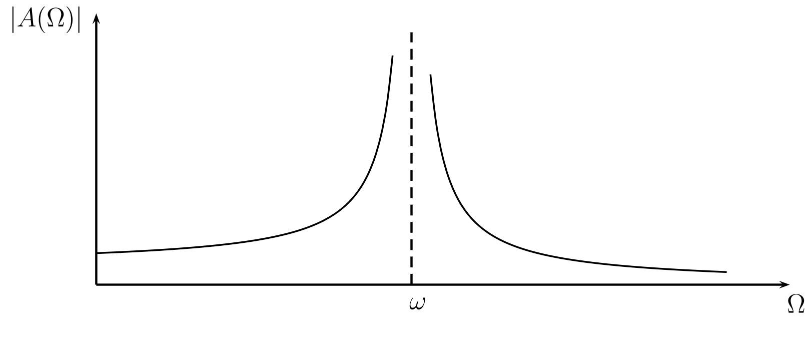

Figure 4.4: Absolute value $|A|$ of the amplitude in the case of an enforced harmonic oscillation.

Therefore the entire solution is given by

$$\boxed{x(t) = \frac{F_0}{m} \frac{1}{\omega^2 - \Omega^2} \cos(\Omega t)}\,. \tag{4.20}$$

The solution contains the following features:

(i) The body oscillates with the same frequency Ω, which it is forced on by the external force.

(ii) The amplitude A is strongly dependent on the external frequency Ω and grows without bound if Ω reaches the natural frequency ω (see Fig. 4.4). This is called the **resonant frequency**.

(iii) Crossing the resonant frequency ω from below ($\Omega < \omega$) involves a change of the sign of A, or in other words a phase shift about π.

4.5 Damped Enforced Oscillations

Now, we will include stokes friction in our considerations. Adding a friction term on the right hand side of Eq. (4.16) leads to the equation of motion for a **damped enforced oscillation**:

$$m\ddot{x} + \gamma\dot{x} + m\omega^2 x = F_0 \cos(\Omega t)\,. \tag{4.21}$$

With the more advanced ansatz

$$x(t) = A\cos(\Omega t - \phi) \tag{4.22}$$

and the use of trigonometric formulas the parameters of the ansatz can be determined:

$$A = \frac{F_0}{\sqrt{m^2(\omega^2 - \Omega^2)^2 + \gamma^2\Omega^2}} \qquad \text{and} \qquad \tan\phi = \frac{\gamma\Omega}{m(\omega^2 - \Omega^2)} \tag{4.23}$$

Due to the damping the absolute value $|A|$ is now bounded as shown in Fig. 4.5 and the phase shift appears smoothly.

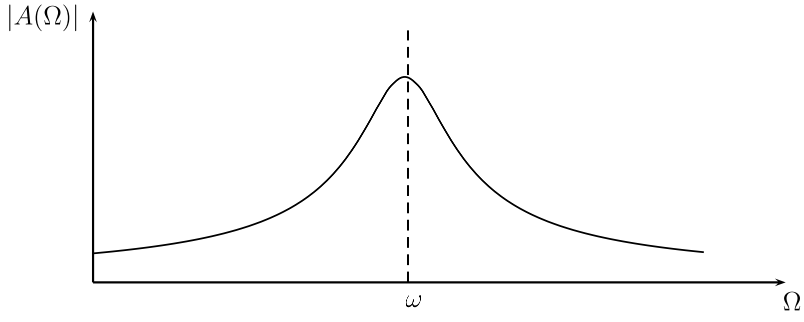

Figure 4.5: Absolute value $|A|$ of the amplitude in the case of an enforced harmonic oscillation with damping.

4.6 Wave Motion

In order to study the motion of waves, disturbances which travel through space and time, we assume a media consisting of mass points with mass m in $d = 1$ dimension and at equilibrium with distance a to each other. They are affiliated by springs, each with the same spring constant D (e.g. in a dense gas with some linear approximated repulsion/attraction between the particles). The equation of motion for mass point n, with position u_n reads

$$\begin{aligned} m\ddot{u}_n &= -D(u_n - u_{n-1}) + D(u_{n+1} - u_n) \\ &= D(u_{n+1} - 2u_n + u_{n-1}) . \end{aligned} \tag{4.24}$$

We are interested in the so called continuum limit, that is $a, m \to 0$, $D \to \infty$ with fixed quantities

$$\mu := \frac{m}{a} \qquad \text{and} \quad Y := aD . \tag{4.25}$$

In labeling the viewed mass points by their equilibrium positions $x = na$ and the definition $\Delta x := a$, we can write

$$\mu \Delta x \, \ddot{u}_n = \frac{Y}{\Delta x}(u(x + \Delta x) - 2u(x) + u(x - \Delta x)) \tag{4.26}$$

or

$$\ddot{u}(x) = \frac{Y}{\mu} \frac{(u(x + \Delta x) - 2u(x) + u(x - \Delta x))}{(\Delta x)^2} \tag{4.27}$$

If we now consider very dense mass point, that is performing the limit $\Delta x \to 0$, the last fraction turns out be the second derivative of $u(x, t)$ with respect to x at a fixed time t. The equation of motion is therefore expressed as

$$\ddot{u}(x, t) - \frac{Y}{\mu} \frac{\partial^2 u(x, t)}{\partial x^2} = 0 . \tag{4.28}$$

which results with the so called **phase velocity** $c = \sqrt{\frac{Y}{\mu}}$ in

$$\boxed{\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x, t) = 0} . \tag{4.29}$$

This equation is called the **wave equation** since it describes the motion of waves. We will consider one special case of solutions, namely the harmonic ones

$$u(x, t) = u_0 \cos(\omega t - kx) \tag{4.30}$$

where ω is the angular frequency of the oscillation at position $x = 0$ the wave number k describes the traveling through space. If we put this ansatz in Eq. 4.29 we obtain a dependency of ω and k, that is

$$-k^2 + \frac{\omega^2}{c^2} = 0 \qquad \Leftrightarrow \qquad \boxed{\omega = ck}\,. \tag{4.31}$$

The latter equation is called the **dispersion relation**. These two equations (4.30) and (4.31) state that the maximum of the wave u is traveling with the phase velocity c, hence its name. The two maxima are separated by the **wave length**

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{c}{f}\,. \tag{4.32}$$

4.7 Superposition and Interference of Waves

The wave equation is a liner equation, which implies that if $u_1(x,t)$ and $u_2(x,t)$ are solutions, than

$$u(x,t) = \alpha u_1(x,t) + \beta u_2(x,t) \tag{4.33}$$

is another solution. Therefore the principle of superposition, we know from forces, applies to waves, too.
In the following, we will consider a fixed position $x = 0$ where two waves $u_1(0,t) = u_1(t) = \cos(\omega_1 t)$ and $u_2(0,t) = u_2(t) = \cos(\omega_2 t)$ are present. The sum $u(t)$ with the aid of $\cos(a) + \cos(b) = 2\cos(\frac{a-b}{2})\cos(\frac{a+b}{2})$ is found to be

$$u(x,t) = \cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\,. \tag{4.34}$$

The implication of this formula, in the case when both frequencies are close, is shown in Fig. 4.6. Therefore, if two frequencies are added, which are not much distinguishable from each other, i.e. $\omega_1 \approx \omega_2$, an oscillation within another one is obtained. This is shown in the formula by the first term, which describes the slow oscillating envelope with frequency $\omega_e = \left|\frac{\omega_1 - \omega_2}{2}\right|$, and the second one causing the oscillations within it. This phenomenon is know as **beat** and is caused by **constructive and reconstructive interference** of the separate waves.

4.8 Doppler Effect

Moved source
We assume a sound source traveling with velocity v_s and emitting waves

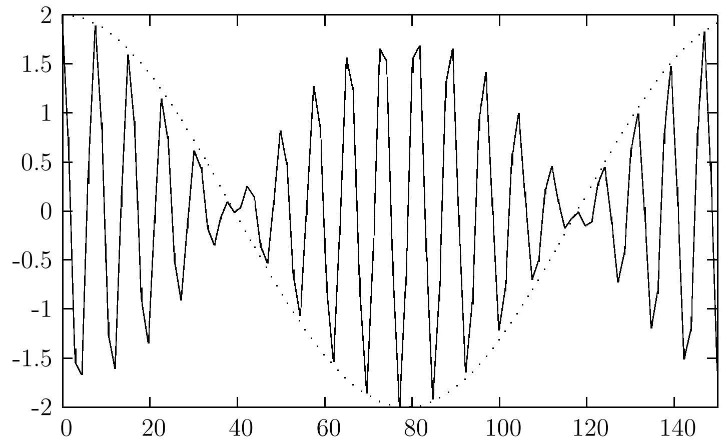

Figure 4.6: Adding two cos-waves with almost the same frequencies leads to beat.

with wavelength $\lambda = c/f$ towards a detector where the velocity of the wave is denoted by c. Within one period of oscillation $T = \frac{\lambda}{c}$ the source travels a distance of $v_s T = v_s \frac{\lambda}{c}$. Thus, the detector measures a wavelength

$$\lambda' = \lambda - v\frac{\lambda}{c} = \lambda\left(1 - \frac{v_s}{c}\right) . \tag{4.35}$$

With $c/f = \lambda$ we obtain

$$\boxed{f' = f\frac{1}{1 - \frac{v_s}{c}}} . \tag{4.36}$$

Therefore the sound of the source seems to be brighter when it is moving towards the detector and darker when it is moving away from it. Note, that for $v_c = c$ the frequency f' becomes infinity. This is called sound barrier.

Moved detector

If the detector is moved with velocity v_d towards a detector and the source stays at rest, than the time T' needed to pass two consecutive maxima is

$$T' = \frac{\lambda}{c + v_d} \tag{4.37}$$

Thus, the frequency $f' = 1/T'$ the detector measures is given by

$$f' = \frac{c}{\lambda}\left(1 + \frac{v_d}{c}\right) \tag{4.38}$$

or with $f = \frac{c}{\lambda}$

$$\boxed{f' = f\left(1 + \frac{v_d}{c}\right)} . \tag{4.39}$$

Part II

Classical Electrodynamics

Chapter 5

Electrostatics

5.1 The Electric Charge

Already the Greek philosophers knew that when rubbing two rods of amber on fur the two rods will repel each other. Since the masses of the rods have not changed, another cause, the **electric charge**, has to be defined. As found by Millikan in 1909, this electric charge is quantized, i.e. every charge Q can be written as

$$Q = n \cdot e, \quad \text{with} \quad n \in \mathbb{Z} \qquad [Q] = \mathrm{C} \tag{5.1}$$

with the **elementary charge**

$$e = 1.602 \cdot 10^{-19}\,\mathrm{C}\,. \tag{5.2}$$

It appears as two different kinds, which are represented by the sign of Q, and for which holds:

(i) Homonymous charges repel each other.

(i) In-homonymous charges attract each other.

5.2 Maxwell's Equations

The fundamental laws in classical mechanics are Newton's axioms (in particular the principle of action). An analog exists in the field of classical electrodynamics - the four Maxwell equations, which are heuristic equations (i.e. the can not be derived from some "higher law"). In order to separate the system from the change which is introduced by measuring it, fields $\vec{E}$ and $\vec{B}$ are introduced to describe what kind of force a test charge would notice in the presence of the system. For completeness, we will list the Maxwell's equations determining these fields and state their meaning while the quantities therein will be discussed afterwards:

(1) **Gauss' law**
Gauss's law identifies the electric charge as he source of an electric field. Therefore the electric flux $\int \vec{E}\,\mathrm{d}\vec{A}$ flowing out of a closed surface S is proportional to the electric charge Q_{in} enclosed in the surface:

$$\oint_S \vec{E}\,\mathrm{d}\vec{A} = \frac{Q_{in}}{\epsilon_0}$$

with the **permittivity of free space**

$$\epsilon_0 = 8.85 \cdot 10^{-12} \frac{\mathrm{C}^2}{\mathrm{Jm}}\,. \tag{5.3}$$

(2) **Gauss' law for magnetism**
Gauss' law for magnetism merely states the absence of magnetic monopoles:

$$\oint_S \vec{B}\,\mathrm{d}\vec{A} = 0$$

(3) **Faraday's law of induction**
Faraday's law of induction (more generally, the law of electromagnetic induction) states that a magnetic field $\vec{B}$ changing in time creates a proportional electro-motive force. The relation between the rate of change of the magnetic flux through the surface S enclosed by a contour C and the electric field along the contour reads:

$$\oint_C \vec{E}\,\mathrm{d}\vec{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{B}\,\mathrm{d}\vec{A}$$

(4) **Ampere's law**
Ampere's law, discovered by Andree-Marie Ampere, relates the magnetic field in a closed loop C to the electric current $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$ passing through the area S enclosed by the loop. It is the magnetic equivalent of Faraday's law of induction:

$$\oint_C \vec{B}\,\mathrm{d}\vec{\ell} = \mu_0 I + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{E}\,\mathrm{d}\vec{A}$$

with the **magnetic constant** or the **permeability of vacuum**

$$\mu_0 = 1.26 \cdot 10^{-6} \frac{\mathrm{J}}{\mathrm{A}^2\mathrm{m}}\,. \tag{5.4}$$

5.3 The Electric Field - Coulomb's Law

In the following, we want to derive the **electric field** $\vec{E}$ which is defined by the force per test charge q:

$$\boxed{\vec{E} = \frac{\vec{F}}{q}} \qquad [E] = \frac{\mathrm{N}}{\mathrm{C}} = \frac{\mathrm{V}}{\mathrm{m}} \tag{5.5}$$

at first for a point charge Q. From Gauss' law this field at a distance r can be calculated when considering a sphere centered around that point charge (see Fig. 5.1). Due to the symmetry, the electric field is constant along the sphere and can therefore be calculated by

$$\oint_S \vec{E}\,\mathrm{d}\vec{A} = \vec{E} \cdot 4\pi r^2 \hat{\vec{r}} = \frac{Q}{\epsilon_0} \qquad \text{or} \qquad \boxed{E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}}. \tag{5.6}$$

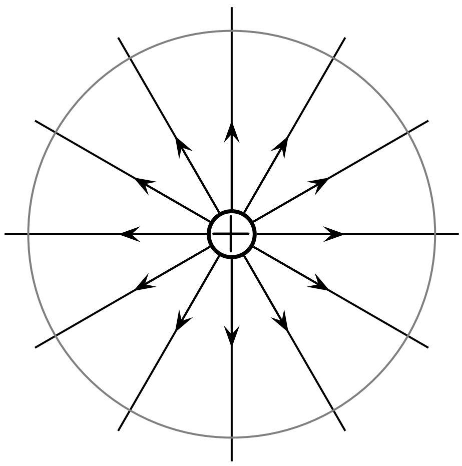

Figure 5.1: Cut through the spherical electric field of a positive point charge and a sphere where the electric field is constant.

This leads to the force between two elementary charges $q_i = Z_i e_i$ with $Z_i = \pm 1$ and $i = 1, 2$ as

$$\vec{F}_{12} = q_2 \vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r_1} - \vec{r_2}|^3}(\vec{r_1} - \vec{r_2}) \qquad \text{or} \quad F_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{(r_{12})^2} \tag{5.7}$$

where $r_{12} = |\vec{r_1} - \vec{r_2}|$ states the distance between the two charges. This result, known as **Coulomb's law**, is in analogy to the gravitational force between two mass points with masses m_1 and m_2.

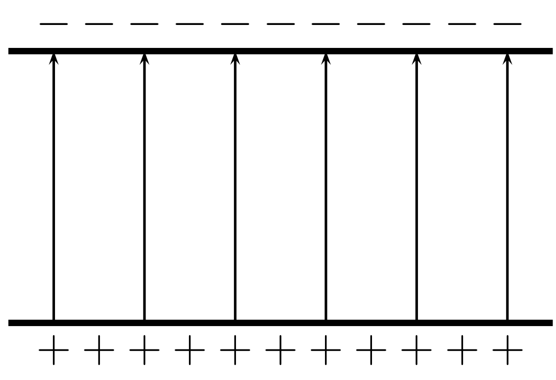

Figure 5.2: Homogeneous Electric Field between two plates with charge $\pm Q$, separated by a distance d.

Now we consider two plates with area A, separated by a distance d, each carrying a charge density $\sigma = \pm\frac{Q}{A}$ (see Fig. 5.2). In order to calculate the electric field between the plates we make again use of the first Maxwell equation.

$$\oint_S \vec{E}\,\mathrm{d}\vec{A} = EA = \frac{Q}{\epsilon_0} \quad \Leftrightarrow \quad \boxed{E = \frac{\sigma}{\epsilon_0}}. \tag{5.8}$$

Thus, we obtain a homogeneous (i.e. constant in space) electric field which corresponds to a homogeneous force

$$|\vec{F}| = q|\vec{E}| = q\frac{\sigma}{\epsilon_0} \tag{5.9}$$

and therefore a motion with constant acceleration.

5.4 Electric Potential, Voltage and Work in an Electric Field

When describing an electric force it is often easier to use the scalar **electric potential** ϕ instead of the vector of the electric field. It is defined by

$$\boxed{\phi(\vec{r}) = -\int \vec{E}\,\mathrm{d}\vec{r}}. \tag{5.10}$$

In applications, however, the derived quantity **voltage** is used, which is defined by the difference of the potential at two points $\vec{r}_1$ and $\vec{r}_2$:

$$U = \Delta\phi = \phi(\vec{r}_2) - \phi(\vec{r}_1) \qquad [U] = \mathrm{V} \tag{5.11}$$

Since the definition of the electric potential is similar to the one of work the latter one is given by

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\,\mathrm{d}\vec{r} = q\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\,\mathrm{d}\vec{r} = qU\,. \tag{5.12}$$

EXAMPLE We reconsider the two plates in Fig. 5.2 and determine the electric field between them as well as the work connected to the transport of charges between the plates.
Integrating the constant electric field $E = \frac{\sigma}{\epsilon_0}$ yields

$$\int_0^d E\,\mathrm{d}x = Ed = U \quad \Leftrightarrow \quad \underline{\underline{E = \frac{U}{d}}}\ . \tag{5.13}$$

Therefore the electric field between two parallel plates can be expressed as a ratio of the voltage between them to their distance d. The work which is gained/needed when an electric charge is transported from one plate to the other can simply be expressed in case of a homogeneous field as

$$\underline{\underline{W}} = qU = \underline{\underline{qEd}}\ . \tag{5.14}$$

5.5 Application: Motion in a Homogeneous Electric Field

We consider a particle of mass m and charge q, which is accelerated due to an (acceleration) voltage U_0. During this process the particle gains kinetic energy, i.e.

$$qU_0 = \frac{1}{2}mv^2 \quad \Leftrightarrow \quad v = \sqrt{\frac{2qU_0}{m}}\,. \tag{5.15}$$

Afterwards the particle enters a parallel plate capacitor charged by another voltage U as depicted in Fig. 5.3. Due to the electric field the particle is accelerated only in y-direction. With $a = \frac{F}{m} = \frac{qE}{m} = \frac{qU}{md}$ the motion is described as

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} vt \\ \frac{qU}{2md}t^2 \end{pmatrix}\,. \tag{5.16}$$

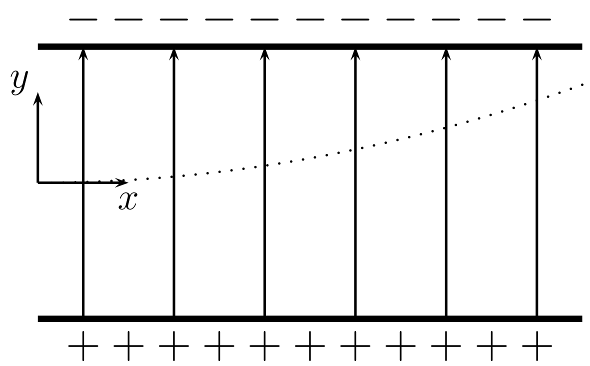

Figure 5.3: If a charged particle enters a parallel plate capacitor it is accelerated in the y-direction. The particle's path forms a parabola.

If we are only interested in the geometrical path we can solve $x = vt$ for $t = x/v$. If we put this in $y(t)$ we get

$$\underline{\underline{y(x) = \frac{1}{2}\frac{qU}{mdv^2}x^2}} \quad . \tag{5.17}$$

5.6 Capacity

The **capacity** is defined as the quotient of charge and voltage:

$$\boxed{C = \frac{Q}{U}} \qquad [C] = \mathrm{F} \tag{5.18}$$

In the case of the two parallel plates of size A which are at distance d, we obtain

$$C = \frac{Q}{U} = \frac{\sigma A}{Ed} = \epsilon_0 \frac{A}{d} \,. \tag{5.19}$$

Imagine now that the capacitor is discharged by bringing small amounts of charge to the other side. Since the voltage will change in this process according to the definition of the capacity the total work is given by

$$W = \int \mathrm{d}W = \int U \mathrm{d}Q = \int \frac{Q}{C}\mathrm{d}Q = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CU^2 \,. \tag{5.20}$$

For a parallel plane capacitor this can be expressed by the electric field:

$$W = \frac{1}{2}\epsilon_0 \frac{A}{d}(Ed)^2 = \frac{1}{2}V\epsilon_0 E^2 \tag{5.21}$$

where V denotes the volume where the electric field is present. This means the energy density is proportional to the electric field squared, which can be derived generally, too.

5.7 Matter in an Electric Field

We reconsider the parallel plate capacitor. Putting a dielectric material between the plates causes an increase in the capacitance with proportion factor ϵ_r, the **relative permittivity** of the material:

$$\epsilon_r = \frac{C_r}{C} \quad \Leftrightarrow \quad \boxed{C_r = \epsilon_r \epsilon_0 \frac{A}{d}}. \tag{5.22}$$

This happens because an electric field polarizes the molecules of the dielectric or orientated existing polar regions (dipoles), producing concentrations of charge on its surfaces that create an electric field opposed (anti-parallel) to that of the capacitor. Therefore, a given amount of charge produces a weaker field E_r between the plates than it would without the dielectric $E = \epsilon_r E_r$, with $\epsilon_f > 1$. This reduces the electric potential, too. Considered in reverse, this argument means that, with a dielectric, a given electric potential causes the capacitor to accumulate a larger charge.

Chapter 6

Electric Direct Current

6.1 Electric Current

As a simple model for the electric current we consider a conductor, which has a density of charge carriers (i.e. electrons with mass $m = m_e$) n. We want to write down the equation of motion for one of this carriers if it is exposed to an electric field E. This particular electron will make collisions with all the other electrons. The time between two collisions will be denoted by τ. These collision processes effectively lead to a friction force, which we will model as Stokes's friction $\gamma v = \frac{m}{\tau} v$. Thus, the equation of motion takes the form

$$m\dot{v}(t) = eE - \frac{m}{\tau} v(t)\,. \tag{6.1}$$

A static (time independent) solution of this problem is given by

$$v = \frac{e\tau}{m} E\,. \tag{6.2}$$

This is the velocity of one carrier. The total velocity of all carriers is $v_{tot} = nv$. Thus the electric current density reads

$$j = ev_{tot} = \frac{ne^2\tau}{m} E = \sigma\, E \tag{6.3}$$

where we defined the **conductivity**

$$\sigma = \frac{ne^2\tau}{m}\,. \tag{6.4}$$

If we are dealing with a homogeneous conductor with length ℓ and cross section A Eq. 6.4 may be integrated to yield:

$$\int j\,\mathrm{d}A = jA = I \quad \text{and} \quad \int E\,\mathrm{d}\ell = E\ell = U \tag{6.5}$$

Thus

$$\boxed{U = \frac{\ell}{\sigma A} I = R\, I}. \qquad [I] = \mathrm{A} \tag{6.6}$$

This relation is known as (the famous) **Ohm's law**. We defined the **resistivity**

$$R = \frac{1}{\sigma}\frac{\ell}{A} = \rho_s \frac{\ell}{A} \qquad [R] = \Omega \tag{6.7}$$

with the so called **specific resistance** $\rho_s = \frac{1}{\sigma}$.

6.2 Kirchhoff's Rules

In electric circuits one often deals with several connected conductors. To calculate for instance the individual currents and voltages or the total resistivity of the system **Kirchhoff's rules** provide a very useful tool.

(i) **Kirchhoff's current law**
Since the electric charge is conserved, at any point in an electrical circuit the charge density does not change with time. Therefore the sum of currents flowing towards that point is equal to the sum of currents flowing away from that point:

$$\sum_k I_k = 0 \tag{6.8}$$

(ii) **Kirchhoff's voltage law**
From energy conservation follows, that the directed sum of the electrical potential differences around a circuit must be zero:

$$\sum_k U_k = 0 \tag{6.9}$$

From there one can e.g. find the rules for the series connection and the parallel connection of resistors (see Fig. 6.1). One finds

(i) **Series connection**

$$R = \sum_k R_k \tag{6.10}$$

(ii) **Parallel connection**

$$R = \frac{1}{\sum_k \frac{1}{R_k}} \tag{6.11}$$

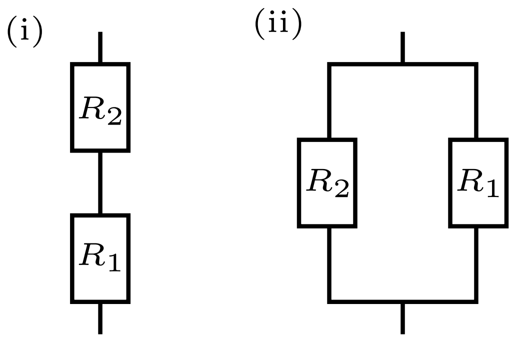

Figure 6.1: Series (i) and parallel (ii) connection of two resistors.

For capacitances the following rules hold

(i) **Series connection**

$$C = \frac{1}{\sum_k \frac{1}{C_k}} \tag{6.12}$$

(ii) **Parallel connection**

$$C = \sum_k C_k \tag{6.13}$$

6.3 Electric Work and Power

We consider again a conductor with resistance R on which ends a voltage U is applied and through which therefore flows a current I. In it energy is dissipated, given by

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = U\frac{\mathrm{d}Q}{\mathrm{d}t} = UI\,. \tag{6.14}$$

Taking into account Ohm's law, i.e $U = RI$ this may be rewritten as

$$\boxed{P = UI = \frac{U^2}{R}}\,. \qquad [P] = \mathrm{W} \tag{6.15}$$

Chapter 7

Electromagnetism

7.1 The Magnetic Field

In the 13th century new navigational instruments were invented using materials which orientate themselves along a certain direction. Since they were not electric charged another field has to be introduced in order to explain the observed force. This magnetic field B as shown by Ampere's law arises from moving charges.
Consider a long wire which carries a current I. By reason of symmetry and on the basis of the second maxwell equation $\int_S \vec{B}\,\mathrm{d}\vec{A} = 0$ the magnetic field has only a tangential component. This can be calculated by the static case of the fourth Maxwell equation (Ampere's law):

$$\int_C \vec{B}\,\mathrm{d}\vec{\ell} = B\,2\pi r = \mu_0 I \quad \Leftrightarrow \quad \boxed{B = \frac{\mu_0 I}{2\pi r}}, \qquad [B] = T \tag{7.1}$$

In order to find the magnetic field inside a coil with length ℓ and number of turns N we again make use of Ampere's law. The left hand side of this equation yields $B\ell$ and for the right hand side we get $\mu_0 N I$. Thus, the magnetic field of a coil is given by

$$\boxed{B = \mu_0 \frac{N}{\ell} I}. \tag{7.2}$$

Thus, we obtain a homogeneous magnetic field inside a coil.

7.2 Lorentz Force

The **Lorentz force** is the force exerted on a charged particle in an electromagnetic field. Due to the electric field, the particle will experience a force of $q\vec{E}$, and due to the magnetic field one of magnitude $q\vec{v} \times \vec{B}$. Combined they give the Lorentz force equation:

$$\boxed{\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B})} \tag{7.3}$$

EXAMPLE We will now consider the motion of a particle with mass m and charged by q in a homogeneous magnetic field $\vec{B}$ perpendicular to the initial velocity of magnitude v.
Since the Lorentz force is perpendicular to the direction of the magnetic field the velocity stays perpendicular to it. Therefore the particle is moving on a circle where the Lorentz force $F_L = qvB$ acts as the centrifugal force The radius of this circle is determined by

$$qvB = \frac{mv^2}{r} \quad \Leftrightarrow \quad \underline{\underline{r = \frac{mv}{qB}}}\ . \tag{7.4}$$

When a wire which is placed in a magnetic field B perpendicular to the direction of its carrying current I the charge carriers will perform similar motions and collide with the atoms of the wire. Thereby the Lorentz force is transfered to a force F acting on the whole wire. Is is determined by

$$F = I\ell B\,. \tag{7.5}$$

EXAMPLE Consider a region of space between the plates of a capacitor where there is an electric field and a perpendicular magnetic field. Show that this is a **velocity selector**.
Imagine a particle of charge q which enters this space from the left. If q is positive, the electric force of magnitude qE points down and the magnetic force of magnitude qvB points up. If the charge is negative, each of these two forces is reversed. The forces balance only for one given velocity:

$$qE = qvB \quad \Leftrightarrow \quad \underline{\underline{v = \frac{E}{B}}}\ . \tag{7.6}$$

7.3 Magnetic Flux and Magnetic Induction

In order to characterize the magnetic field by one scalar value, the **magnetic flux** Φ is defined via

$$\Phi = \int \vec{B}\,\mathrm{d}\vec{A} = BA, \qquad [\Phi] = \mathrm{Wb} \tag{7.7}$$

where the last equality holds only for a homogeneous field. Thus, Maxwell's third equation (Faraday's law of induction) may be rewritten as

$$\int_C \vec{E}\,\mathrm{d}\vec{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \vec{B}\,\mathrm{d}\vec{A} = -\dot{\Phi}\,. \tag{7.8}$$

Since the left hand side of this equations is a voltage we finally obtained the **induced voltage**

$$\boxed{U_{\mathrm{ind}} = -\dot{\Phi}}\,. \tag{7.9}$$

7.4 Self-Inductance

We consider an empty coil with

$$B = \mu_0 \frac{N}{\ell} I\,. \tag{7.10}$$

If the current I through the coil changes with time we get an induced voltage

$$U_{\mathrm{ind}} = -\frac{\mathrm{d}}{\mathrm{d}t}(BA)N = -\frac{\mathrm{d}}{\mathrm{d}t}\left(A\mu_0\frac{N}{\ell}IN\right) = -\left(\mu_0\frac{AN^2}{\ell}\right)\dot{I}\,. \tag{7.11}$$

If we define the **inductance** to be

$$L = \mu_0\frac{AN^2}{\ell}, \qquad [L] = \mathrm{H} \tag{7.12}$$

we get

$$\boxed{U_{\mathrm{ind}} = -L\dot{I}}\,. \tag{7.13}$$

The last equation is **Lenz's law**:

The induced current produced in the conductor always flows in such a direction that the magnetic field it produces will oppose the change that produces it.

7.5 Matter in a Magnetic Field

Like in the case of the electric field, the magnetic field induces or orientates magnetic moments within matter. The resulting change of the strength of the magnetic field is characterized by the **relative permeability**

$$\boxed{\mu_r = \frac{B_r}{B}} \tag{7.14}$$

where B_r is the magnetic field in the presence of matter and B without it. However, materials exhibit quite different reactions in the presence of a magnetic field. They are sorted in the following categories:

- In **diamagnetic** materials magnetic moments are induced. From Lenz's law they are therefore orientated in a way that the external magnetic field is weakened. Thus, the relative permeability possesses small values $0 < \mu_r < 1$.
- **Paramagnetism** is observed if existing magnetic moments within matter only influence each other very weakly. Since the moments are not induced, it is favorable to align them with the external field. Therefore the external field is enhanced and therefore the relative permeability $\mu_r > 1$ is obtained.
- **Ferromagnetic** matter, like iron, will keep the magnetic field after the external one is not present any more. That means they posses magnetic moments of their own, too. But the strong coupling between them prevent the magnetic field to vanish again. However, if the applied magnetic field is very weak, the process can still be reversed and Eq. (7.14) can be applied. In this case their relative permeability is very large $\mu_r \gg 1$ (e.g. $\mu_{r,\mathrm{iron}} \sim 5000$).

7.6 Applications: Electromagnetic Machines

In this section we will consider some important examples for application, namely the electric motor, electric generator and the transformer.

Electric generator
We consider a rectangular coil of N turns and with area A in an uniform magnetic field B. In the simplest picture of a generator this rectangular coil is now (due to an external force) rotated with angular velocity ω. Thus, the effective flux through the rectangular coil is $\Phi = NBA\cos(\omega t)$ and we get for the induced voltage

$$U(t) = -\dot{\Phi} = NBA\omega\sin(\omega t)\,. \tag{7.15}$$

Thus the generator converts mechanical energy into electric energy. However, in contrast to the kind of voltage discussed in chapter 6, the voltage obtained varies harmonic with time. Its corresponding current is therefore an **alternating-current (AC)**. In order to characterize it usually the frequency and the **rms-value** of the voltage is stated. The latter quantity is defined as the voltage which would give the same power dissipation at a resistor. For a harmonic time-dependence this differs from the maximum voltage only by a factor $1/\sqrt{2}$. For the generator we obtain

$$U_{\text{rms}} = \frac{NBA\omega}{\sqrt{2}} \,. \tag{7.16}$$

Electric motor
To convert electric energy into mechanical energy we have to apply an external AC-voltage to the coil, i.e.

$$U_{\text{ext}} = U_0 \sin(\omega t) \,. \tag{7.17}$$

Due to that external AC-voltage the coil will rotate with angular velocity ω.

EXAMPLE As an example we want to determine the angular velocity ω of an electric motor without load when applying an AC-voltage $U_{\text{ext}} = U_0 \cos(\Omega t)$ as well as the magnetic field B necessary. The motor consists of a quadratic frame with edge length ℓ inside the magnetic field B.
If the frame would rotate inside the external magnetic field the flux through it would be given by

$$\Phi = B\ell^2 \sin(\alpha)$$

where α is the angle between the (edges of the) frame and the magnetic field. Kirchhoff's voltage law states therefore

$$U_0 \cos(\Omega t) = U_{\text{ext}} = -U_{\text{ind}} = \dot{\Phi} = B\ell^2 \dot{\alpha} \cos(\alpha)$$

This equation is fulfilled for a motion with constant angular velocity $\underline{\underline{\omega}} = \dot{\alpha} = \underline{\underline{\Omega}}$ and if the magnetic field matches the applied voltage $\underline{\underline{B = \frac{U_0}{\Omega \ell^2}}}$.

Transformer
If a time-varying voltage U_P is applied to the primary winding of N_P turns, a current will flow in it producing a magneto-motive force. Just as an electromotive force drives current around an electric circuit, so does the magnetomotive force an magnetic flux through a magnetic circuit. The primary magneto-motive force produces a varying magnetic flux Φ_P in the core, and induces a back electro-motive force in opposition to U_P in the secondary circuit. In accordance with Faraday's law of induction, the voltage induced

across the primary and secondary winding is proportional to the rate of change of flux:

$$U_P = N_P \frac{\mathrm{d}\Phi_P}{\mathrm{d}t} \quad \text{and} \quad U_S = N_S \frac{\mathrm{d}\Phi_S}{\mathrm{d}t} \tag{7.18}$$

where

- U_P and U_S are the voltages across the primary winding and secondary winding.
- N_P and N_S are the numbers of turns in the primary winding and secondary winding.
- $\mathrm{d}\Phi_P/\mathrm{d}t$ and $\mathrm{d}\Phi_S/\mathrm{d}t$ are the derivatives of the flux with respect to time of the primary and secondary winding.

Saying that the primary and secondary windings are perfectly coupled is equivalent to saying that $\Phi_P = \Phi_S$. Substituting and solving for the voltages shows that:

$$\frac{U_P}{U_S} = \frac{N_P}{N_S} \tag{7.19}$$

Hence in an ideal transformer, the ratio of the primary and secondary voltages is equal to the ratio of the number of turns in their windings, or alternatively, the voltage per turn is the same for both windings. The ratio of the currents in the primary and secondary circuits is inversely proportional to the turns ratio. This leads to the most common use of the transformer: To convert electrical energy at one voltage to energy at a different voltage by means of windings with different numbers of turns. In a practical transformer, the higher-voltage winding will have more turns and smaller conductor cross-section, than the lower-voltage windings.

Chapter 8

Electric Oscillations and Electric Waves

8.1 The RL-Circuit

An AC through a coil
We consider a coil which is directly connected to an alternating current $I(t) = I_0 \sin(\omega t)$. From the definition of the self-inductance we obtain the voltage as

$$U = L\dot{I} = LI_0\omega\cos(\omega t) \tag{8.1}$$

Therefore the voltage obtains its maximum one quarter of the period before the current. We say the current follows the voltage by 90°, and in analogy to Ohm's resistance the **inductive reactance**

$$X_L = \frac{U_0}{I_0} = \frac{U_{\text{rms}}}{I_{\text{rms}}} = \omega L \tag{8.2}$$

is defined.

Opening of an RL-Circuit (Discharging)
Now, we add a resistor to the circuit and consider a DC-source (see Fig. 8.1a). In the first case, we assume that the upper switch S in Fig. 8.1a is opened (respectively the lower switch S' is closed) at time $t = 0$, i.e. the voltage source is disconnected from the circuit, but an initial current I_0 remains. Kirchhoff's voltage law in combination with the definition of the self-inductance and Ohm's law states for this case

$$0 = I(t)R + L\dot{I}(t)\,. \tag{8.3}$$

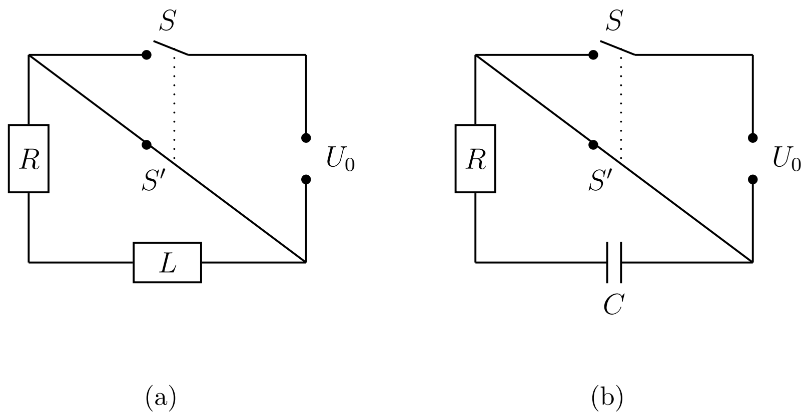

Figure 8.1: (a) RL-circuit and (b) RC-circuit.

The solution with $I(0) = I_0$ reads in this case

$$\boxed{I(t) = I_0 \mathrm{e}^{-\frac{R}{L}t}}\,. \tag{8.4}$$

EXAMPLE In order to determine the energy stored in a coil with self-inductance L, N turns and length ℓ, through which initially a current I_0 flows, we consider the coil connected to a resistor R.
Since the time dependent power dissipated at the resistor amounts to

$$P(t) = U_R(t)I(t) = R\big(I(t)\big)^2 = R{I_0}^2\mathrm{e}^{-2\frac{R}{L}t}$$

the total energy that is dissipated can be calculated by

$$\underline{\underline{E_I}} = \int P(t)\mathrm{d}t = R{I_0}^2\int_0^\infty \mathrm{e}^{-2\frac{R}{L}t}\mathrm{d}t = R{I_0}^2\frac{L}{2R}\int_{-\infty}^0 \mathrm{e}^{t'}\mathrm{d}t' = \underline{\underline{\frac{1}{2}L{I_0}^2}}$$

where the substitution $t' = -2\frac{R}{L}t$ was performed to evaluate the integral. Since initially only a magnetic field $B = \mu_0\frac{N}{\ell}I_0$, generated by the coil, was present this energy has to be stored in the magnetic field. We can connect the energy directly to the stored magnetic field by

$$\underline{\underline{E_B}} = E_I = \frac{1}{2}L{I_0}^2 = \frac{1}{2}\cdot\mu_0\frac{N^2A}{\ell}\cdot\frac{B^2\ell^2}{\mu^2N^2} = \underline{\underline{\frac{1}{2}\frac{V}{\mu_0}B^2}}$$

where in the last step the volume V where the magnetic field is present was introduced.

Closing of an RL-Circuit (Charging)
Now, we change the position of the switches, i.e. the DC-voltage source is connected to the circuit. Therefore Kirchhoff's voltage law states with the definition of the self-inductance and Ohm's law:

$$U_0 = I(t)R + L\dot{I}(t) \tag{8.5}$$

The solution for $I(0) = 0$ of this differential equation is given by

$$\boxed{I(t) = \frac{U_0}{R}\left(1 - \mathrm{e}^{-\frac{R}{L}t}\right)}\,. \tag{8.6}$$

8.2 The RC-Circuit

An AC through a capacitor
At first, we consider a capacitor which is directly connected to an AC-voltage $U(t) = U_0 \sin(\omega t)$. From the definition of the capacity we get the current

$$I = C\dot{U} = CU_0\omega\cos(\omega t)\,. \tag{8.7}$$

Therefore the voltage follows the current by 90° and the **capacitive reactance**

$$X_C = \frac{U_0}{I_0} = \frac{U_{\mathrm{rms}}}{I_{\mathrm{rms}}} = \frac{1}{\omega C} \tag{8.8}$$

can be defined in analogy to Ohm's resistance.

Opening of an RC-Circuit (Discharging)
Now, we consider the right circuit in Fig. 8.1, where a capacitor is disconnected from a DC-voltage source, but can discharge via a resistor R. From Kirchhoff's voltage law we get with Ohm's law and the definition of the capacity with $I = -\dot{Q}$:

$$\frac{1}{C}Q(t) + R\dot{Q}(t) = 0 \tag{8.9}$$

The solution to this homogeneous differential equation with $Q(0) = Q_0$ is given by

$$\boxed{Q(t) = Q_0\mathrm{e}^{-\frac{t}{RC}}}\,. \tag{8.10}$$

This states an exponential decay with **time constant** $\tau = RC$.

Closing of an RC-Circuit (Charging)
Finally, we consider a closed switch S where a capacitor is charged by a DC-voltage source via a resistor R. Kirchhoff's voltage reads now

$$U_0 = R\dot{Q}(t) + \frac{1}{C}Q(t)\,. \tag{8.11}$$

The solution of this equation with $Q(0) = 0$ is given by

$$\boxed{Q(t) = CU_0\left(1 - \mathrm{e}^{-\frac{t}{RC}}\right)}\,. \tag{8.12}$$

8.3 Electric Resonant Circuits

An RLC circuit (also known as a resonant circuit or a tuned circuit) is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel. We will consider only the series connection. Since the total voltage amounts to be zero and with $I = \dot{Q}$ we obtain

$$0 = \frac{Q}{C} + RI + L\dot{I} = \frac{1}{C}Q + R\dot{Q} + L\ddot{Q}\,. \tag{8.13}$$

This equation is analogous to the mechanical equation for damped oscillations $Dx + \gamma\dot{x} + m\ddot{x} = 0$ if one makes the following replacements:

$$x \leftrightarrow Q \qquad m \leftrightarrow L \qquad \gamma \leftrightarrow R \qquad D \leftrightarrow \frac{1}{C} \tag{8.14}$$

We will first consider the case $R = 0$. Eq. (8.13) simplifies to

$$\frac{1}{C}Q(t) + L\ddot{Q}(t) = 0\,. \tag{8.15}$$

Since differential equation for $Q(t)$ describes a harmonic oscillation it has the solution

$$Q(t) = Q_0 \sin(\omega t + \phi_0) \qquad \text{with} \quad \omega = \frac{1}{\sqrt{LC}}\,. \tag{8.16}$$

For the general case (i.e. $R \neq 0$) we can take once the time derivative of Eq. (8.13) to obtain

$$0 = \frac{1}{C}I(t) + R\dot{I}(t) + L\ddot{I}(t)\,. \tag{8.17}$$

As discussed in sect. 4.4 the solution (for $R^2 < \frac{4L}{C}$) is given by

$$I(t) = A\mathrm{e}^{-\gamma t}\cos(\omega t + \phi) \quad \text{with} \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}, \quad \gamma = \frac{R}{2L}\,. \tag{8.18}$$

In this solution the magnetic field inside the coil and the electric field inside the capacitor are oscillatory generated while the other field is annihilated. Similar to the discussion in sect. 4.6 such an oscillatory disturbance will propagate in space. Therefore this device can be used to radiate **electromagnetic waves**.

In order to compensate the losses by the resistor an additional AC-source U_0 with frequency Ω can be put in series into the circuit. Then the system would correspond to the damped enforced oscillation discussed in section 4.5. The phase shift between the applied voltage and the current in the system can then be expressed by the effective resistances

$$\tan\phi = \frac{X_L - X_C}{R} \tag{8.19}$$

while the solution for the current is given by

$$I = \frac{U_0}{Z}\cos(\Omega t - \phi) \tag{8.20}$$

with the **impedance**

$$Z = \sqrt{(X_L - X_C)^2 + R^2}\,. \tag{8.21}$$

8.4 Summary: Electronics

	Resistor	Capacitor	Inductor
Connection to the voltage	$R = \frac{U}{I}$	$C = \frac{Q}{U}$	$L = \frac{\lvert U\rvert}{\lvert\dot{I}\rvert}$
Connection to geometry	$R = \rho_s \frac{\ell}{A}$	$C = \epsilon_0\epsilon_r \frac{A}{d}$	$L = \mu_0\mu_r \frac{N^2 A}{\ell}$
Series connection	$R_{\text{tot}} = \sum_k R_k$	$\frac{1}{C_{\text{tot}}} = \sum_k \frac{1}{C_k}$	$L_{\text{tot}} = \sum_k L_k$
Parallel connection	$\frac{1}{R_{\text{tot}}} = \sum_k \frac{1}{R_k}$	$C_{\text{tot}} = \sum_k C_k$	$\frac{1}{L_{\text{tot}}} = \sum_k \frac{1}{L_k}$
Reactance in an AC-circuit	R	$X_C = \frac{1}{\omega C}$	$X_L = \omega L$
Energy (DC-case) dissipated / stored	$W_R = \frac{1}{R}U^2\,t$	$W_C = \frac{1}{2}CU^2$	$W_L = \frac{1}{2}LI^2$
Energy density of the generated fields		$w_C = \frac{1}{2}\epsilon_0\epsilon_r E^2$	$w_L = \frac{1}{2}\frac{1}{\mu_0\mu_r}B^2$