**Pre-Semester Physics - Rigid Body** **Summer 2010**

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**Summary**
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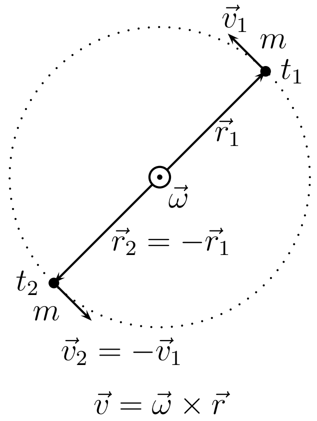
# 1 Rotation of one particle




a) no MC: at $t_1$: $\vec{p}_1 = m\vec{v}_1$
at $t_2$: $\vec{p}_2 = m\vec{v}_2 = -m\vec{v}_1$

but $\vec{r}_2 = -\vec{r}_1$

- Angular Momentum (AM)

$$\boxed{\vec{L} = \vec{r} \times \vec{p}} = m\vec{r} \times \vec{v} = m(r^2\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r})$$

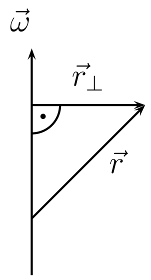
where the last term vanishes if the point of origin lies in the plane of the motion.

- Moment of Torque

$$\boxed{\vec{M} = \vec{r} \times \vec{F}}$$

- Equation of Motion (EoM)

$$\dot{\vec{L}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{r} \times \vec{v}) = m\vec{v} \times \vec{v} + m\vec{r} \times \vec{a} = \vec{r} \times \vec{F} = \vec{M}$$



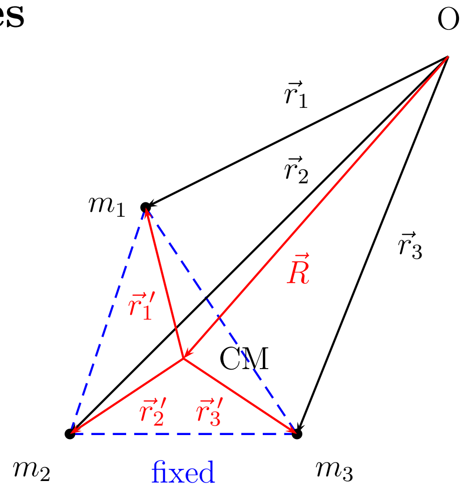

- EC:

$$\underline{\underline{E_r}} = \frac{1}{2}m\vec{v}^{\,2} = \frac{1}{2}m(\vec{\omega} \times \vec{r})^2 = \frac{1}{2}m\omega^2 {r_\perp}^2 = \underline{\underline{\frac{1}{2}\Theta\omega^2}}$$

which is called the Rotational Energy with the Moment of Inertia

$$\Theta = m{r_\perp}^2 \,.$$

# 2 Motion of multiple particles but with rigid connections




I) Center of mass (CM) : $\vec{R} = \frac{\sum_j m_j \vec{r}_j}{\mathcal{M}}$

with $\mathcal{M} = \sum_j m_j$

II) Relative coordinate : $\vec{r}_j{}' = \vec{r}_j - \vec{R}$

a) Energy

$$E = \frac{1}{2}\sum_j m_j (\dot{\vec{r}}_j)^2 = \frac{1}{2}\sum_j m_j \dot{\vec{R}}^2 + \frac{1}{2}\sum_j m_j (\dot{\vec{r}}_j{}')^2 + \dot{\vec{R}}\sum_j m_j \dot{\vec{r}}_j{}'$$

The last term vanishes since

$$\sum_j m_j \dot{\vec{r}}_j{}' \overset{II}{=} \sum_j m_j \dot{\vec{r}}_j - \sum_j m_j \dot{\vec{R}} \overset{I}{=} \mathcal{M}\dot{\vec{R}} - \mathcal{M}\dot{\vec{R}} = \vec{0}$$

$$\Rightarrow \quad E = \frac{1}{2}\mathcal{M}\dot{\vec{R}}^2 + \frac{1}{2}\sum_j m_j (\dot{\vec{r}}_j{}')^2$$

For an axis through the CM where $\dot{\vec{r}}_j{}' = \vec{\omega} \times \vec{r}_j{}'$ holds:

$$E = \frac{1}{2}\mathcal{M}\dot{\vec{R}}^2 + \frac{1}{2}\omega^2 \sum_j m_j (\vec{r}_{\perp,j}{}')^2$$

$$= \underbrace{\frac{1}{2}\mathcal{M}\dot{\vec{R}}^2}_{E_{\text{kin}}} + \underbrace{\frac{1}{2}\Theta\omega^2}_{E_{\text{rot}}}$$

with the total mass $\mathcal{M} = \sum_j m_j$

and the moment of inertia $\Theta = \sum_j m_j (r_{\perp,j}{}')^2$

b) Equation of motion

$$\vec{P} = \mathcal{M}\dot{\vec{R}} \qquad \underline{\underline{\vec{L}}} = \sum_j \vec{L}_j \overset{AM}{=} \sum_j m_j \vec{\omega}(r_{\perp,j}{}')^2 = \underline{\underline{\Theta\vec{\omega}}}$$

$$\dot{\vec{P}} = \vec{F}_{tot} \qquad \underline{\underline{\dot{\vec{L}}}} = \sum_j \dot{\vec{L}}_j = \sum_j \vec{M}_j = \underline{\underline{\vec{M}}}$$

c) Moment of inertia

- Continuous body

$$\underline{\underline{\Theta}} = \sum_j (r_{\perp,j}{}')^2 m_j \rightarrow \int_{\mathcal{M}} (r_\perp{}')^2 \mathrm{d}m = \underline{\underline{\int_V \rho (r_\perp{}')^2 \mathrm{d}V}}$$

where in the last step the DENSITY $\rho = \frac{\mathrm{d}m}{\mathrm{d}V}$ was introduced.

- Arbitrary axis parallel to one through the CM and at a distance $d$:

$$\tilde{\vec{r}}_j = \vec{r}_j' + \vec{d}$$

The new moment of inertia can be related to the one for the axis through the CM:

$$\begin{aligned} \underline{\underline{\Theta}} &= \sum_j m_j (\tilde{\vec{r}}_{\perp,j})^2 = \sum_j m_j (\vec{r}_{\perp,j}{}')^2 + d^2 \sum_j m_j + 2d \underbrace{\sum_j m_j \vec{r}_{\perp,j}{}'}_{=0} \\ &= \underline{\underline{\Theta_{CM} + \mathcal{M} d^2}} \end{aligned}$$

This is known as the Parallel-Axis Theorem.