KIT - International Department GmbH

Pre-Semester Physics - Exercises   Summer 2010

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1. Exercise:

A rigid body with mass $m = 1\,\mathrm{kg}$ is fixed at one point O. However, it can rotate around this point with a moment of inertia of $\Theta = 5\,\mathrm{kg\,m^2}$. Its center of mass is at a distance of $\ell = 2\,\mathrm{m}$ from the point O. Gravity is acting on the body with $g = 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$

(a) What is the moment of torque $|\vec{M}|$ at its vertical position below O.

(b) Determine the moment of torque at an angle $\alpha = 30°$ with respect to the vertical.

(c) State the equation of motion. (Do not attempt solve it!)

(d) Determine the angular velocity ω at the vertical position if the body was released at an angle $\alpha = 30°$.

(e) What would be the velocity v' of the rigid body (again formerly released at the angle $\alpha = 30°$) after it elastically hits another body with mass $m_2 = 100\,\mathrm{kg}$ at the vertical position? (The second mass should formerly be at rest.)

Solution:

(a) Since the gravitational force F_G is perpendicular to the direction the body can move it is

$$\underline{\underline{|\vec{M}|}} = |\vec{r} \times \vec{F_G}| = \ell F_G \sin 0 = \underline{\underline{0}} \,.$$

(b) At a finite angle the force is not perpendicular any more. Therefore only its projection perpendicular to the radial coordinate contributes to the moment of torque:

$$\underline{\underline{|\vec{M}|}} = \ell F_G \sin\alpha = mg\ell \sin\alpha = \underline{\underline{10\,\mathrm{Nm}}}$$

(c) From the formula above for the moment of torque at finite angles the equation of motion reads

$$\vec{M} = \dot{\vec{L}} \qquad \Rightarrow \qquad \underline{\underline{mg\ell\sin\alpha}} = -\Theta\dot{\omega} = \underline{\underline{-\Theta\ddot{\alpha}}} \,.$$

(d) The conservation of energy gives (compare with exercise 4 on sheet 4)

$$E_{\text{pot}} = E_{\text{rot}} \quad \Rightarrow \quad mg\ell(1-\cos\alpha) = \frac{1}{2}\Theta\omega^2$$

$$\Rightarrow \quad \underline{\underline{\omega}} = \sqrt{\frac{2mg\ell(1-\cos\alpha)}{\Theta}} = \underline{\underline{1.04\,\frac{1}{\text{s}}}} \,.$$

(e) Energy conservation states

$$E_{\text{rot}} \stackrel{!}{=} E'_{\text{rot}} + E'_{\text{kin}} \quad \Rightarrow \quad \frac{1}{2}\Theta\omega^2 = \frac{1}{2}\Theta\frac{v'^2}{\ell^2} + \frac{1}{2}m_2 v_2'^2$$

where v' and v_2' denotes the velocity of the rigid and the other body after the collision. The former one was already connected to the angular velocity by $v' = \omega'\ell$. Since the force at the collision is perpendicular to the initial velocity this perpendicular component satisfies the momentum conservation law:

$$mv = m\omega\ell = mv' + m_2 v_2' \quad \Rightarrow \quad v_2' = \frac{m}{m_2}(\omega\ell - v')$$

Inserting it in the energy conservation the final velocity of the rigid body can be obtained:

$$\begin{aligned}
\Theta m_2 \ell^2 \omega^2 &= \Theta m_2 v'^2 + \ell^2 m^2(\omega^2\ell^2 - 2\omega\ell v' + v'^2) \\
\Rightarrow \quad 0 &= v'^2 - 2\omega\ell\frac{\ell^2 m^2}{\ell^2 m^2 + \Theta m_2}v' + \frac{\ell^2 m^2 - \Theta m_2}{\ell^2 m^2 + \Theta m_2}\ell^2\omega^2 \\
\Rightarrow \quad \underline{\underline{v'}} &= \frac{\ell^2 m^2}{\ell^2 m^2 + \Theta m_2}\omega\ell - \sqrt{\frac{\ell^4 m^4}{(\ell^2 m^2 + \Theta m_2)^2} - \frac{\ell^2 m^2 - \Theta m_2}{\ell^2 m^2 + \Theta m_2}\omega^2\ell^2} \\
&= \frac{\ell^2 m^2 - \Theta m_2}{\ell^2 m^2 + \Theta m_2}\omega\ell \\
&= \underline{\underline{-2.0\,\frac{\text{m}}{\text{s}}}}
\end{aligned}$$

where the negative solution was chosen since the positive one represents the situation without a collision.

2. Exercise:

Consider two masses m separated by a rod with length ℓ and edge length d. However, the coordinates along the edge length should be neglected against the one along the length ℓ. The system is rotated around an axis perpendicular to the rod. At first neglect the mass m_r of the uniform rod.

(a) What is the moment of inertia of the two masses regarding the center of mass?

(b) Determine the moment of inertia if it is rotated around the position of one mass.

(c) Calculate the moment of inertia of the rod if it would be rotated around its center of mass.

(d) What is the moment of inertia regarding the center of mass if the mass of the rod is considered?

(e) Which value does the moment of inertia posses regarding the position of one mass if again the mass m_r of the rod is considered?

Solution:

(a) Taking the definition of the moment of inertia as the sum of the moments of inertia of all particles the result is given by

$$\underline{\underline{\Theta}} = m\left(\frac{\ell}{2}\right)^2 + m\left(\frac{\ell}{2}\right)^2 = \underline{\underline{\frac{1}{2}m\ell^2}} \ .$$

(b) Since only one mass is rotating at the length ℓ:

$$\underline{\underline{\Theta_m = m\ell^2}}$$

(c) From the integral definition of the moment of inertia the result is given by

$$\begin{aligned} \underline{\underline{\Theta_r}} &= \int \rho \vec{r}^2 \,\mathrm{d}V = \rho \int_{-\frac{d}{2}}^{\frac{d}{2}} \mathrm{d}x \int_{-\frac{d}{2}}^{\frac{d}{2}} \mathrm{d}y \int_{\frac{\ell}{2}}^{\frac{\ell}{2}} (x^2+y^2+z^2)\mathrm{d}z \\ &\approx \frac{m_r}{d^2\ell} \int_{-\frac{d}{2}}^{\frac{d}{2}} \mathrm{d}x \int_{-\frac{d}{2}}^{\frac{d}{2}} \mathrm{d}y \int_{\frac{\ell}{2}}^{\frac{\ell}{2}} z^2 \,\mathrm{d}z = \frac{m_r}{d^2\ell} \cdot d \cdot d \cdot \frac{2}{3}\frac{\ell^3}{8} = \underline{\underline{\frac{1}{12}m_r\ell^2}} \ . \end{aligned}$$

where in the third step the contribution perpendicular to the length is neglected.

(d) Again the final total moment of inertia is given by the sum over its parts:

$$\underline{\underline{\Theta_{cm}}} = m\frac{\ell^2}{4} + m\frac{\ell^2}{4} + \frac{1}{12}m_r\ell^2 = \underline{\underline{\frac{6m+m_r}{12}\ell^2}}$$

(e) With the parallel axis theorem the total moment of inertia is determined by

$$\underline{\underline{\Theta_{tot}}} = \Theta_{cm} + (2m+m_r)\left(\frac{\ell}{2}\right)^2 = \underline{\underline{\frac{3m+m_r}{3}\ell^2}} \ .$$

3. Exercise:

Consider a body with mass m attached to a rope which is wound around a spool with radius r and moment of inertia Θ. The spool can rotate around the axis O through its center of mass. Neglect friction and assume that the rope pulls without slipping.

(a) State the equation of motion of the system.

(b) What is the acceleration a of the mass m?

(c) Find the tension force F_T in the rope.

Solution:

(a) The balance of moment of torque for the spool reads

$$\underline{\underline{r\,F_T = \Theta\dot{\omega}}}$$

while the balance of forces for the mass is given by

$$\underline{\underline{ma = mg - F_T}}$$

where the acceleration is measured downwards.

(b) Substituting $a = \dot{\omega}r$ in the second equation and inserting the first one in it determines the angular acceleration:

$$m\dot{\omega}r^2 = mgr - \Theta\dot{\omega} \qquad \Rightarrow \qquad \dot{\omega} = \frac{mgr}{\Theta + mr^2} = \frac{g}{r}\frac{1}{1 + \frac{\Theta}{mr^2}}$$

This means the acceleration is given by

$$\underline{\underline{a = g\frac{1}{1 + \frac{\Theta}{mr^2}}}}$$

(c) With these results the first equation of motion determines the tension force to

$$\underline{\underline{F_T}} = \frac{g\Theta}{r^2}\frac{1}{1 + \frac{\Theta}{mr^2}} = \underline{\underline{\frac{mg}{1 + \frac{mr^2}{\Theta}}}}$$