KARLSRUHE INSTITUTE OF TECHNOLOGY - INTERNATIONAL DEPARTMENT GMBH

Pre-Semester 2010 - Physics Course - Extra Tutorial

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Solution 13
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1. **RC-Circuit**

Consider a capacitor C which is connected in series to two parallel connected resistors R_1 and R_2. Initially, the charge Q on (one of) the capacitor's plate, is $Q(t=0) = Q_0$.

(a) Find the charge $Q(t)$ in the capacitor as a function of time t if initially it is $Q(t = 0) = Q_0$. What is the time constant τ?

(b) What is the initial (total) current I_0 through the resistors? What is the initial voltage U_0 across the capacitor?

(c) If after time $t_{1/2} = 20.79\,\text{ms}$ the charge Q has dropped to 50% of its initial value, what is τ?

(d) In the situation of (1c): What is R_1 if $R_2 = R_1$, $Q_0 = 0.3\,\text{C}$ and $U_0 = 200\,\text{V}$?

Solution:

(a) The total resistance of the parallel connection of the two resistors is

$$R = \frac{1}{1/R_1 + 1/R_2}.$$

Kirchhoff's rule gives the "equation of motion"

$$0 = Q/C + IR = Q/C + \dot{Q}R.$$

With the initial condition $Q(t = 0) = Q_0$ the solution is

$$Q(t) = Q_0 e^{-t/\tau}$$

where the time constant is

$$\tau = RC = \frac{C}{1/R_1 + 1/R_2}.$$

(b) The initial current is

$$I_0 = \dot{Q}(t = 0) = -\frac{Q_0}{RC}.$$

The initial voltage is

$$U_0 = Q_0/C.$$

Thus (up to signs) Ohm's law is satisfied:

$$|I_0| = |U_0|/R.$$

(c) $t_{1/2}$ is defined by the equation

$$Q(t_{1/2}) = \frac{1}{2}Q_0 \quad \Leftrightarrow \quad e^{-t_{1/2}/\tau} = \frac{1}{2} \quad \Leftrightarrow \quad -t_{1/2}/\tau = \ln\frac{1}{2} = -\ln 2$$

$$\Leftrightarrow \tau = \frac{t_{1/2}}{\ln 2} = 30.0\,\text{ms}.$$

(d)

$$\begin{aligned} U_0 = Q_0/C \quad &\Rightarrow \quad C = Q_0/U_0, \\ \tau = RC \quad &\Rightarrow \quad R = \tau/C = \tau U_0/Q_0, \\ R = R_1/2 \quad &\Rightarrow \quad R_1 = 2R = 2\tau U_0/Q_0 = 40\,\Omega. \end{aligned}$$

2. LR-Circuit

Consider a coil with inductance L and a resistor R connected in series. Assume that for $t < 0$ they were additionally connected in series to a voltage source with voltage U_0 driving a stationary current I_0. At $t = 0$ the voltage source is removed.

(a) What is I_0 (in terms L, R, and U_0)?

(b) Find the current $I(t)$ through the resistor as a function of time t. What is the time constant τ?

(c) What is the voltage drop $U_R(t)$ across the resistor at time t?

(d) Sketch $U_R(t)$ and mark U_0 and τ in the diagram.

Solution:

(a) Ohm's law: $I_0 = U_0/R$.

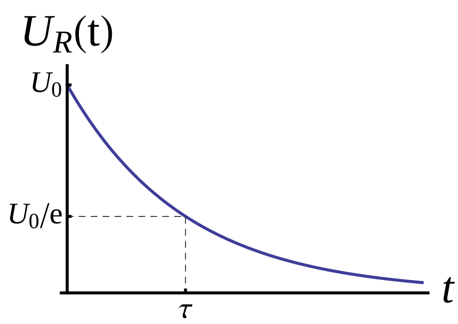
(b) Kirchhoff's rule: $= IR + L\dot{I}$.
Solution with initial condition $I(t = 0) = I_0$ is $I(t) = I_0 e^{t/\tau}$ with time constant

$$\tau = L/R.$$

(c) Again Ohm's law:

$$U_R(t) = RI(t) = RI_0 e^{-t/\tau} = U_0 e^{-t/\tau}.$$

(d) Plot:



3. LC-Circuit

A parallel-plate capacitor with capacitance C, which has been charged by a voltage U_0, is connected at $t = 0$ to an ideal coil with inductance L. No voltage sources are present (hence, the initial current is $I(t = 0) = 0$).

(a) What is the maximal charge Q_0 in the capacitor?

(b) Give the charge $Q(t)$ in the capacitor as a function of time t.

(c) What is the maximal current I_0 and at what time t_0 is it reached, $|I(t_0)| = I_0$, the first time?

(d) Sketch $Q(t)$ as well as the current $I(t) = \dot{Q}(t)$ and mark Q_0, I_0, t_0 and the period T in the diagrams.

(e) Assume that, in case the capacitor is empty, current oscillates with frequency $f_0 = T^{-1} = 80\,\text{kHz}$. What would be the frequency f_r in case the capacitor is filled with a dielectric material with relative permittivity $\epsilon_r = 16$?

(f) Assume now that a resistor is connected in series to L and C. Give a *qualitative* sketch of $I(t)$ for this case. (Do not calculate anything!)

Solution:

(a) Initially, the capacitor is maximally charged: $Q_0 = CU_0$.

(b) Kirchhoff's rule gives

$$0 = Q/C + L\dot{I} = Q/C + L\ddot{Q} \quad \Leftrightarrow \quad \ddot{Q} + \frac{1}{LC}Q = 0.$$

Differential equation well known, describes harmonic oscillation. Initial conditions read

$$Q(t=0) = Q_0, \qquad \dot{Q}(t=0) = 0.$$

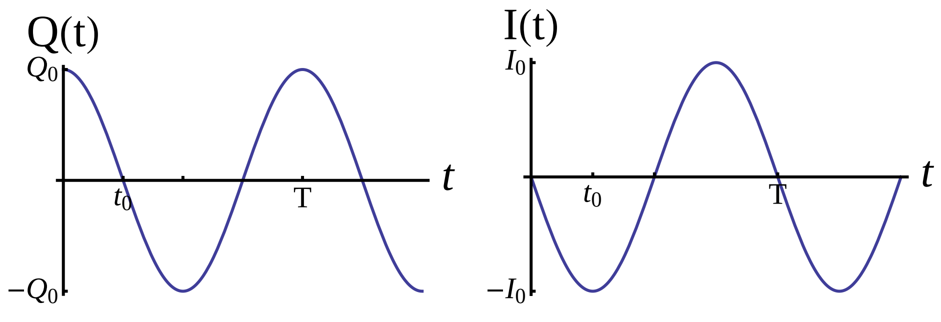
The solution is

$$Q(t) = Q_0 \cos(\omega t) \text{ with } \omega = \frac{1}{\sqrt{LC}}.$$

(c) The current is $I(t) = \dot{Q}(t) = -Q_0\omega \sin(\omega t)$. It is maximal the first time, when $\sin(\omega t) = 1$ the first time, that means for

$$\omega t_0 = \frac{\pi}{2} \quad \Leftrightarrow \quad t_0 = \frac{\pi}{2\omega}.$$

(d) Plots:



(e) For an empty capacitor the frequency is

$$80\,\text{kHz} = f_0 = \frac{1}{2\pi}\frac{1}{\sqrt{LC_0}}.$$

For a filled capacitor it is

$$f_r = \frac{1}{2\pi}\frac{1}{\sqrt{LC_r}} = \frac{1}{2\pi}\frac{1}{\sqrt{L\epsilon_r C_0}} = \frac{f_0}{\sqrt{\epsilon_r}} = 20\,\text{kHz}.$$

(f) Plot: