Karlsruhe Institute of Technology - International Department GmbH

Pre-Semester 2010 - Physics Course - Extra Tutorial

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Solution 14
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1. Generator

In a homogeneous magnetic field B a rectangular metal frame is rotated around one of its (central) axes with constant angular velocity ω. Each of the edges which are *parallel* to the rotation axis is a wire with length ℓ, circular cross section with (small) diameter d, and resistivity $\rho_{\|}$. Each of the edges which are *perpendicular* to the rotation axis is a wire with length s, (small) cross section A_{cs}, and resistivity $\rho_{\perp}$.

Initially, the frame is oriented perpendicular to the magnetic field.

(a) Calculate the total resistance R of the metal frame.

(b) What voltage $U_{\text{ind}}(t)$ is induced in the metal frame? (Neglect self-induction effects.)

(c) Find the current $I(t)$ which runs through the frame in terms of B, ω, ℓ, s, and R.

(d) What electric power $P_{\text{el}}(t)$ is dissipated in the frame?

(e) What force $F_L(t)$ is exerted by the magnetic field on each of the edges parallel to the rotation axis?

(f) What is the moment of torque $M(t)$ which is necessary to keep the frame rotating with constant angular velocity ω?

(g) What is the mechanical power $P_{\text{mech}}(t)$ provided by this moment of torque?

Solution:

(a) The resistance of each of the *perpendicular* wires is $R_{\perp} = \rho_{\perp} s / A_{\text{cs}}$. The cross section of the *parallel* wires is a circle with radius $d/2$, that means area $\tilde{A} = \pi d^2/4$. Hence, the resistance of each of the parallel wires is $R_{\|} = \rho_{\|}\ell/\tilde{A} = 4\rho_{\|}\ell/(\pi d^2)$. The total resistance is

$$R = 2R_{\perp} + 2R_{\|} = 2\frac{\rho_{\perp} s}{A_{\text{cs}}} + 8\frac{\rho_{\|}\ell}{\pi d^2}.$$

(b) If α is the angle between the frame and the magnetic field, then the perpendicular component is $B_{\perp} = B\sin\alpha$, and the magnetic flux is $\Phi = B_{\perp}A = B\ell s\sin\alpha$.
It is time-dependent because the frame rotates with angular velocity $\dot{\alpha}(t) = \omega$, starting with $\alpha(t=0) = \pi/2$. Hence,

$$\alpha(t) = \omega t + \frac{\pi}{2}.$$

Noting $\sin(\pi/2 + x) = \cos(x)$ the time-dependent flux is

$$\Phi(t) = B\ell s\sin(\frac{\pi}{2} + \omega t) = B\ell s\cos(\omega t).$$

According to Faraday's law of induction, the voltage

$$U_{\text{ind}}(t) = -\dot{\Phi}(t) = B\ell s\omega\sin(\omega t)$$

is induced.

(c)

$$\text{Ohm's law: } I(t) = \frac{U_{\text{ind}}(t)}{R} = \frac{B\ell s\omega}{R}\sin(\omega t).$$

(d)

$$P_{\text{el}}(t) = U_{\text{ind}}(t)^2/R = \frac{B^2\ell^2 s^2\omega^2}{R}\sin(\omega t)^2.$$

(e)

$$\text{Lorentz force: } F_L(t) = I(t)\ell B = \frac{B^2\ell^2 s\omega}{R}\sin(\omega t).$$

(f) Perpendicular component of force: $F_\perp = F_L\cos(\alpha)$.
With $\alpha(t) = \pi/2 + \omega t$ and $\cos(x + \pi/2) = \sin(x)$ we get

$$M(t) = 2\cdot\frac{s}{2}F_\perp(t) = sF_L(t)\cos(\alpha(t)) = -\frac{B^2\ell^2 s^2\omega}{R}\sin(\omega t)^2.$$

(g)

$$P_{\text{mech}}(t)M(t)\omega = -\frac{B^2\ell^2 s^2\omega^2}{R}\sin(\omega t)^2.$$

Not caring about signs, we like this result because $|P_{\text{el}}(t)| = |P_{\text{mech}}(t)|$.

2. Physical Pendulum

Consider a physical pendulum which consists of a rod with mass m_r and length ℓ (and negligible lateral extent) and of a point mass with mass m_p which is attached in the middle of the rod. The pendulum may rotate around an axis which goes through one of the rod's ends. Gravity g is acting.

(a) Calculate the moment of inertia Θ of the pendulum.

(b) In case the pendulum makes an angle ϕ with its equilibrium (vertical) position, what is the moment of torque acting on it?

(c) Write down the equation of motion for $\phi(t)$ and approximate it for small angles.

(d) Assuming the physical pendulum to be initially at rest and to make a (small) angle ϕ_0 with the vertical, give the solution $\phi(t)$.

(e) Sketch $\phi(t)$ and mark the initial angle ϕ_0 and the period T in the diagram.

(f) Sketch what would change (qualitatively) if we took, say, (weak) Stokes friction into account?

Note: The moment of inertia of a rod (mass m_r, length ℓ) with respect to a perpendicular axis through its center of mass is $\Theta_{\text{cm}} = \frac{1}{12}m_r\ell^2$.

Solution:

(a) **Rod:** $\Theta_{\text{cm}} = \frac{1}{12}m_r\ell^2$
Parallel-axis theorem: $\Theta_r = \Theta_{\text{cm}} + m_r\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}m_r\ell^2$
Point: $\Theta_p = m_p\left(\frac{\ell}{2}\right)^2$
Total: $\Theta = \Theta_r + \Theta_p = \frac{1}{3}m_r\ell^2 + \frac{1}{4}m_p\ell^2$

(b) Gravity effectively acts on center of mass, total mass $\boxed{m = m_r + m_p}$: $F_g = mg$.
Perpendicular component $F_\perp = F_g \sin\phi = mg\sin\phi$.
Moment of torque (with correct sign):

$$M = -F_\perp \frac{\ell}{2} = -\frac{mg\ell}{2}\sin\phi$$

(c) Equation of motion $\boxed{\dot{L} = M}$ with $L = \Theta\dot{\phi}$ gives

$$\Theta\ddot{\phi} = M - \frac{mg\ell}{2}\sin\phi \quad \Leftrightarrow \quad 0 = \ddot{\phi} + \frac{mg\ell}{2\Theta}\sin\phi.$$

For small angles we have $\sin\phi \approx \phi$, and thus

$$0 = \ddot{\phi} + \frac{mg\ell}{2\Theta}\phi.$$

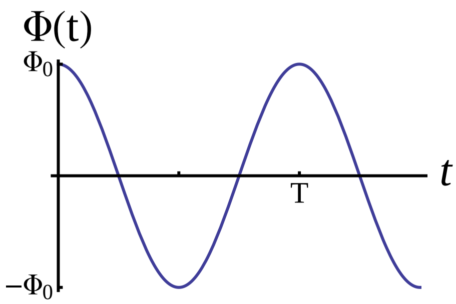
(d) For the harmonic oscillation we use the general ansatz $\phi(t) = A\cos(\omega t + \delta)$ with

$$\omega = \sqrt{\frac{mg\ell}{2\Theta}}.$$

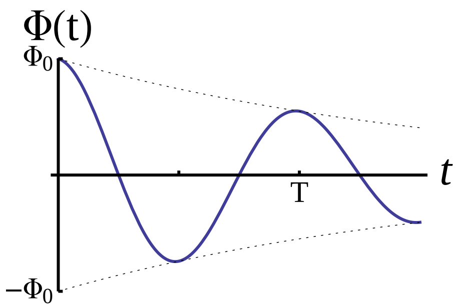
With the initial conditions $\phi(t=0) = \phi_0$, $\dot{\phi}(t=0) = 0$ the solution reads

$$\phi(t) = \phi_0 \cos(\omega t).$$

(e) Plot:



(f) Plot:



3. Falling Stone

A stone is shot in horizontal (x-)direction with velocity $v_{x0} = 10\,\mathrm{m/s}$, starting at $x(t = 0) = 0$ and in a height $y(t=0) = h = 5\,\mathrm{m}$. Gravity is acting in negative y-direction, $g = 10\,\mathrm{m/s^2}$.

(a) Give the coordinates $x(t)$, $y(t)$ as function of time.

(b) At what time T does the stone hit the ground? What is its range s?

(c) What is the stone's velocity v_f shortly before the impact? What angle α does it make with the ground?

Solution:

(a) Uniform motion in x-direction: $x(t) = v_{x0}t$
Constant acceleration in y-direction: $y(t) = -\frac{1}{2}gt^2 + h$

(b) Impact on ground:

$$0 = y(T) = -\frac{1}{2}gT^2 + h \quad \Leftrightarrow \quad T = \sqrt{\frac{2h}{g}} = 1\,\mathrm{s},$$

$$\text{Range: } s = x(t)v_{x0}T = v_{x0}\sqrt{\frac{2h}{g}} = 10\,\mathrm{m}$$

(c) Impact velocity:

$$v_{xf} = \dot{x}(T) = v_{x0} = 10\,\frac{\mathrm{m}}{\mathrm{s}},$$

$$v_{yf} = \dot{y}(T) = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh} = -10\,\frac{\mathrm{m}}{\mathrm{s}},$$

$$v_f = \sqrt{v_{xf}^2 + v_{yf}^2} = 14.1\,\frac{\mathrm{m}}{\mathrm{s}},$$

$$\tan\alpha = \frac{v_{xf}}{v_{yf}} \quad \Rightarrow \quad \alpha = \arctan\frac{v_{xf}}{v_{yf}} = -45^\circ.$$