Pre-Semester 2010 - Physics Course - Extra Tutorial

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Solution 3
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1. Pendulum

A pendulum of length $l = 80\,\text{cm}$ with a bob of mass $m = 0.6\,\text{kg}$ is released from rest at initial angle of θ_0 with the vertical. At the bottom of the swing, the speed of the bob is $v_b = 2.8\,\text{m/s}$.

(a) What was the initial angle θ_0 of the pendulum?

Solution: The simplest way to determine θ_0 is via energy considerations: The total energy of the system is $E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = \frac{1}{2}mv^2 + mgz$. We choose a coordinate system such that the bottom of the swing is at $z = 0$. Then, the angle θ with the vertical satisfies $z = l(1 - \cos\theta)$. Initially, the bob is at rest and $E_{\text{kin}} = 0$. At the bottom of the swing, $E_{\text{pot}} = 0$. Due to energy conservation $\frac{1}{2}mv_b^2 = E_{\text{tot}} = mgz_0 = mg(1 - \cos\theta_0) \Rightarrow \cos\theta_0 = 0.51 \Rightarrow \theta_0 \approx 59°$.

(b) What angle θ' does the pendulum make with the vertical when the speed of the bob is $v' = 1.4\,\text{m/s}$?

Solution: Now energy conservation reads $\frac{1}{2}mv_b^2 = E_{\text{tot}} = mgz' + \frac{1}{2}mv'^2 = mg(1 - \cos\theta_0) + \frac{1}{2}mv'2 \Rightarrow \cos\theta' = 1 - \left(v_b^2 - v'^2\right)/(2gl) \approx 0.63 \Rightarrow \theta' \approx 51°$.

2. Loop

Consider a bob of mass m gliding "down" a frictionless slide which contains a loop of radius R. The ball starts at rest at a height $h \geq 2R$ above the bottom of the loop.

(a) Compute the ball's speed v_{top} at the top of the loop.

Solution: The total energy is $E_{\text{tot}} = \frac{1}{2}mv^2 + mgz$. Since in the beginning we have $z = h$ and $v = 0$, it is $E_{\text{tot}} = mgh$. At the top of the loop, $z = 2R$, energy conservation gives $mgh = \frac{1}{2}mv_{\text{top}}^2 + mg\;2R \Rightarrow v_{\text{top}} = \sqrt{2g(h - 2R)}$

(b) What is the path which the bob describes in the case $h = 2R$?

Solution: In this case, the bob reaches the top of the loop with a velocity $v_{\text{top}} = 0$ (that means, without gravity it would just stay there). Due to its weight the bob then falls downward along a straight line toward the bottom of the loop.

(c) What is the minimal height h_{min} from which the bob needs to start in order to make it through the loop?

Solution: To have the bob performing a circular motion of radius R with (instantaneous) velocity v, there has to be a centripetal force $F_{\text{cp}} = mv^2/R$ acting on it which points towards the circle's center. Where does it come from? It is the sum of the normal force F_N exerted by the slide and the radial component $F_{g,r}$ of the weight (the tangential component $F_{g,t}$ changes the velocity $|\vec{v}|$).

The slide takes care that the distance of the bob to the circle's center can never be $> R$. If $F_{g,r} < F_{\rm cp}$, then $F_N = F_{\rm cp} - F_{g,r} > 0$. However, since the bob is not fixed to the slide, the normal force exerted by the slide cannot point away from circle's center $\Rightarrow F_N \leq 0$. In the case $F_{g,r} > F_{\rm cp}$ the net force $F_{g,r} + F_N$ towards the center is larger than the one needed ($F_{\rm cp}$) to keep the bob on the R-circle. The bob is pulled away from the slide. Hence, we require $F_{g,r} < F_{\rm cp}$.

If we denote the position of the bob along the circle by the angle φ between the horizontal and the line connecting bob and circle's center, then $z = R(1 + \sin\varphi)$, $F_{g,r} = mg\sin\varphi$, and due to energy conservation $v^2 = 2g(h - z) = 2gR(h/R - 1 - \sin\varphi)$. Then the condition for the bob to stay on the slide is

$$mg\sin\varphi \leq mv^2/R = 2mg(h/R - 1 - \sin\varphi) \Rightarrow h/R \geq 1 + \frac{3}{2}\sin\varphi.$$

This is satisfied *for all* φ if and only if

$$h/r \geq \frac{5}{2} \Rightarrow h_{\rm max} = \frac{5}{2}R.$$

(In fact, we just have proven that it would have been sufficient to require $F_{g,r} < F_{\rm cp}$ only at the top of the loop.)

3. **Spring**

A block of mass m is dropped onto the top of a vertical spring whose force constant is k. The block is released from a height h above the top of the spring.

(a) What is the maximum kinetic energy $E_{\rm kin,max}$ of the block?

Solution: In this problem there are 3 contributions to the total energy $E_{\rm tot} = E_{\rm grav} + E_{\rm kin} + E_{\rm comp}$:

- $E_{\rm grav} = mgz$, the potential energy due to gravity. Coordinate z is chosen such that the top of the unrelaxed spring is at $z = 0$ and the initial position of the block is $z = h$. Thus, if the block compresses the spring by $s > 0$, its position is $z = -s < 0$, and $E_{\rm grav} < 0$.
- $E_{\rm kin} = \frac{1}{2}mv^2$, the kinetic energy of the block.
- $E_{\rm comp} = \frac{1}{2}ks^2$, the potential energy of the compressed spring. The spring is initially relaxed, its compression then is $s = 0$.

Since the total energy is conserved we may calculate it in the *initial state*, $z = h$, where the block is at rest, $v = 0$, and the spring is not compressed, $s = 0$: $E_{\rm tot} = E_{\rm grav} = mgh$.

As long as the block does not touch the spring it holds $z > 0$, $s = 0$, and thus $E_{\rm kin} \leq E_{\rm tot} = mgh$.

Let us consider the case $z < 0$. Then $s = -z$ and $E_{\rm kin} = E_{\rm tot} - E_{\rm grav} - E_{\rm comp} = mg(h - z) - \frac{1}{2}Dz^2$. The maximal kinetic energy is reached at $z = -mg/D$ (just solve $\mathrm{d}E_{\rm kin})/\mathrm{d}z = 0$), where it assumes the value $E_{\rm kin,max} = mgh + \frac{1}{2}m^2g^2/D > E_{\rm tot}$ (remember that $E_{\rm grav} < 0$!).

(b) What is the maximum compression $s_{\rm max}$ of the spring?

Solution: The spring is maximally compressed when the block has reached its reversal point $z = -s_{\text{max}}$ where $0 = E_{\text{kin}} = mg(h + s_{\text{max}}) - \frac{1}{2}Ds_{\text{max}}^2$. This quadratic equation has two solutions one of which is negativ. The positive one is

$$s_{\text{max}} = \frac{mg}{D}\left(1 + \sqrt{1 + 2\frac{hD}{mg}}\right).$$

(c) At what compression s is the block's kinetic energy half its maximum value?

Solution: We have to solve

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2}E_{\text{kin,max}} \\ \Leftrightarrow mg(h+s) - \frac{1}{2}Ds^2 &= \frac{1}{2}mgh + \frac{1}{4}m^2g^2/D. \end{aligned}$$

The two solutions of this quadratic equation are

$$s = \frac{mg}{D}\left(1 \pm \sqrt{1 + \frac{D}{mg}\left(h - \frac{1}{2}\frac{mg}{D}\right)}\right).$$

4. 2D Collision

A particle with mass m_1 has initial speed v_0. It collides with a second particle with mass m_2 that is at rest, and is deflected through an angle θ_1. Its speed after the collision is v. The second particle recoils, and its velocity makes an angle θ_2 with the initial direction of the first particle.

(a) Show that

$$\tan\theta_2 = -\frac{v\sin\theta_1}{v_0 - v\cos\theta_1}.$$

Solution: Momentum conservation *parallel* and *perpendicular to the initial direction* of the first particle reads,

$$\begin{aligned} m_1v_0 &= m_1v\cos\theta_1 + m_2v_2\cos\theta_2 &\Rightarrow m_2v_2\sin\theta_2 &= -m_1v\sin\theta_1, &(1)\\ 0 &= m_1v\sin\theta_1 + m_2v_2\sin\theta_2 &\Rightarrow m_2v_2\cos\theta_2 &= m_1\left(v_0 - v\cos\theta_1\right), &(2) \end{aligned}$$

where $v_2 = |\vec{v}_2|$ is the final speed of the second particle. Since $\tan x = \sin x/\cos x$, deviding (1) and (2) gives

$$\tan\theta_2 = -\frac{v\sin\theta_1}{v_0 - v\cos\theta_1}.$$

(b) Show that if the collision is elastic and $m_1 = m_2$, then $v = v_0\cos\theta_1$.

Solution: In elastic collisions the kinetic energy is conserved, thus

$$\frac{1}{2}m_1v_0^2 = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v_2^2 \quad \Rightarrow v_2^2 = v_0^2 - v^2.$$

Using $\tan x = \sin x / \cos x$, $\cos^2 x + \sin^2 x = 1$, and the result of (a), we obtain

$$\sin^2 \theta_2 = \frac{\sin^2 \theta_2}{\cos^2 \theta_2 + \sin^2 \theta_2} = \frac{\tan^2 \theta_2}{1 + \tan^2 \theta_2} = \frac{v^2 \sin^2 \theta_1}{v_0^2 + v^2 - 2 v_0 v \cos \theta_1}.$$

And, hence, the square of (2) is

$$m_1 v^2 \sin^2 \theta_1 = m_2 v_2^2 \sin^2 \theta_2 = m_1^2 \left(v_0^2 - v^2\right) \sin^2 \theta_2 = m_1^2 \left(v_0^2 - v^2\right) \frac{v^2 \sin^2 \theta_1}{v_0^2 + v^2 - 2 v_0 v \cos \theta_1}$$
$$\Rightarrow 2 \left(v^2 - v_0 v \cos \theta_1\right) = 0 \Rightarrow v = v_0 \cos \theta_1.$$