Karlsruhe Institute of Technology - International Department GmbH

Pre-Semester 2010 - Physics Course - Extra Tutorial

Stéphane Ngo Dinh
stephane.ngodinh@kit.edu

Solution 4
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1. **Totally inelastic collision - mathematical pendulum**

   Consider a point mass M attached to a *massless* rod with length l which is itself fixed at the ceiling (for short: a "mathematical pendulum"). A bullet of mass m moves horizontally with velocity v toward the point mass. After the impact both, bullet and point mass, stick together.

   (a) What is the total (linear) momentum shortly before the impact?

   (b) What is the velocity v' of the combined object shortly after the impact?

   (c) What is the kinetic energy shortly after the impact and how much energy Q has been converted into heat?

   (d) What is the maximal angle α_{max} to the vertical by which the rod is deflected?

   Solution:

   (a) Only the bullet moves: $p = mv$.

   (b) Momentum conservation gives

   $$mv = (m+M)v' \Rightarrow v' = \frac{m}{m+M}v.$$

   (c) The kinetic energy before the impact is

   $$E_{\text{kin}} = \frac{1}{2}mv^2,$$

   after the collision it is

   $$E'_{\text{kin}} = \frac{1}{2}(m+M)v'^2 = \frac{1}{2}\frac{m^2}{m+M}v^2.$$

   The kinetic energy lost is

   $$Q = E_{\text{kin}} - E'_{\text{kin}} = \frac{1}{2}\frac{mM}{m+M}v^2.$$

   (d) The maximal angle is reached if the complete energy E'_{kin} is converted into potential energy $E_{\text{pot}} = mgl(1-\cos\alpha)$:

   $$E_{\text{pot,max}} = E'_{\text{kin}} \Rightarrow mgl(1-\cos\alpha_{\text{max}}) = \frac{1}{2}\frac{m^2}{m+M}v^2$$

   $$\Rightarrow \alpha_{\text{max}} = \arccos\left[1 - \frac{v^2}{2gl}\frac{m}{m+M}\right].$$

2. **Totally inelastic collision - physical pendulum**

Now we adress the similar, more realistic problem where the rod is not massless. In fact, now the pendulum may have any arbitrary shape. For this, we take all angular quantities (like angular momentum) with respect to the (fixed) axis of rotation. We denote the pendulum's moment of inertia by Θ, its mass by M and the distance of its center of mass to the axis by s.

Again, this pendulum is hit by the same, sticky, bullet at a point which is located directly below the axis, at a distance l.

(a) What is the total angular momentum shortly before the impact?

(b) What is the moment of inertia Θ' of the combined object?

(c) What is its angular velocity ω' shortly after the impact?

(d) What is the kinetic energy shortly after the impact and how much energy Q has been converted into heat?

(e) Does the combined object have the same center of mass as the original pendulum? Where is it now? What is its new distance s' to the axis?

(f) What is the maximal angle $\alpha_{\max}$ to the vertical by which the center of mass is deflected?

(g) Just out of curiosity: What is the moment of inertia Θ'_{cm} of the combined object with respect to an axis through its *center of mass* parallel to the original axis? (*advanced!*)

Solution:

(a) Only the bullet moves: $L = lmv$.

(b) The total moment of inertia is the sum $\Theta' = \Theta + \Theta_b$ where the moment of inertia of the bullet is $\Theta_b = ml^2$.

(c) Conservation of angular momentum gives

$$lmv = \Theta'\omega' \Rightarrow \omega' = \frac{lmv}{\Theta'}.$$

(d) The kinetic energies shortly before and after the impact are

$$E_{\text{kin}} = \frac{1}{2}mv^2 \quad \text{and} \quad E'_{\text{kin}} = \frac{1}{2}\Theta'\omega'^2 = \frac{1}{2}\frac{l^2m^2v^2}{\Theta'}.$$

The energy converted into heat thus is

$$Q = E_{\text{kin}} - E'_{\text{kin}} = \frac{1}{2}mv^2\left(1 - \frac{ml^2}{\Theta'}\right) = \frac{1}{2}mv^2\frac{\Theta}{\Theta'}.$$

(e) Generally, the position of the center of mass is given by $\vec{r}_{\text{cm}} = \left(\sum_i m_i\vec{r}_i\right) / \left(\sum_i m_i\right)$. If one adds a new mass point m at position $\vec{r}_b$ (like the bullet in our case), the new center of mass is located at $\vec{r}^{\,\prime}_{\text{cm}} = \left(m\vec{r}_b + \sum_i m_i\vec{r}_i\right) / \left(m + \sum_i m_i\right)$.

Our situation is especially simple because the origin of the coordinate system (the hinge), the old center of mass and the added mass point are all located on the same line ($\vec{r}_{\text{cm}} \parallel \vec{r}_b$). In this case the new center of mass also lies on this line ($\vec{r}_{\text{cm}}^{\,\prime} \parallel \vec{r}_{\text{cm}}$). But its distance to the hinge has changed:

$$s' = |\vec{r}_{\text{cm}}^{\,\prime}| = \frac{ml + Ms}{m + M}.$$

(f) In order to calculate the potential energy (but not the kinetic energy) one may identify a rigid body with a mass point, where the entire mass is concentrated in the center of mass of the body:

$$E_{\text{pot,max}} = E'_{\text{kin}} \Rightarrow (m+M)gs'(1-\cos\alpha_{\text{max}}) = \frac{1}{2}\frac{l^2m^2v^2}{\Theta + ml^2}$$
$$\Rightarrow \alpha_{\text{max}} = \arccos\left[1 - \frac{1}{2g(ml+Ms)}\frac{l^2m^2v^2}{\Theta + ml^2}\right].$$

(g) Since the hinge has a distance s' to the center of mass of the combined body, the moments of inertia are related via the parallel-axis theorem:

$$\Theta' = \Theta'_{\text{cm}} + (m+M)s'^2 = \Theta'_{\text{cm}} + \frac{(ml+Ms)^2}{m+M} \Rightarrow \Theta'_{\text{cm}} = \Theta' - \frac{(ml+Ms)^2}{m+M}.$$