Pre-Semester 2010 - Physics Course - Extra Tutorial

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Solution 6
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1. Spring

A point mass attached to a (massless) spring with force constant D is initially displaced by x_0 from its equilibrium position (the spring is, say, stretched). It is released at time $t = 0$.

(a) State the equation of motion (for $x(t)$) and solve it.

(b) At which time t_1 does the oscillator (the spring with mass point) return to its initial state the first time? How would you name such a time?

(c) What is the velocity $v(t)$ at time t?

(d) What are the kinetic and potential energies $E_{\text{kin}}(t)$ and $E_{\text{pot}}(t)$ at time t? Compute their sum E_{tot}.

Solution:

(a) Since $F = -Dx$ the equation of motion reads $m\ddot{x} + Dx = 0$. Its solution is $x(t) = x_0 \cos(\omega t)$ with $\omega\sqrt{D/m}$.

(b) The mass point reaches its original position when $\omega t_1 = 2\pi \Rightarrow t_1 = 2\pi/\omega$. Of course, this is the period of oscillation.

(c) $v(t) = \dot{x}(t) - x_0\omega \sin(\omega t)$

(d) The kinetic energy is $E_{\text{kin}}(t) = \frac{1}{2}mv(t)^2 = \frac{1}{2}m\omega^2 x_0^2 \sin(\omega t)^2 = \frac{1}{2}Dx_0^2 \sin(\omega t)^2$. The potential energy is $E_{\text{pot}}(t) = \frac{1}{2}Dx(t)^2 = \frac{1}{2}Dx_0^2 \cos(\omega t)^2$. While both oscillate, their sum $E_{\text{tot}}(t) = E_{\text{kin}}(t) + E_{\text{pot}}(t) = \frac{1}{2}Dx_0^2$ is constant (using $\sin^2 x + \cos^2 x = 1$). It equals the *initial potential energy*.

2. Weakly Damped Harmonic Oscillator

We now assume that the oscillator from the previous exercise moves in a liquid and experiences Stokes friction with coefficient γ. We assume that this damping is weak, $\gamma^2 < 4Dm$.

(a) State the new equation of motion (again for $x(t)$).

(b) Consider the function

$$x(t) = Ae^{-\lambda t}\cos(\omega t + \phi) \quad \text{with} \quad \lambda = \frac{\gamma}{2m} \quad \text{and} \quad \omega = \sqrt{\frac{D}{m} - \frac{\gamma^2}{4m^2}}.$$

Compute its derivatives $\dot{x}(t)$, $\ddot{x}(t)$ and show that it solves the equation of motion.

Solution:

(a) The friction force is $F_s = -\gamma v = -\gamma \dot{x}$ which yields the equation of motion

$$m\ddot{x} + \gamma\dot{x} + Dx = 0.$$

(b) Function and derivatives are

$$\begin{aligned}
x(t) &= Ae^{-\lambda t}\cos(\omega t + \phi),\\
\dot{x}(t) &= Ae^{-\lambda t}\left[-\lambda\cos(\omega t + \phi) - \omega\sin(\omega t + \phi)\right],\\
\ddot{x}(t) &= Ae^{-\lambda t}\left[\left(\lambda^2 - \omega^2\right)\cos(\omega t + \phi) + 2\lambda\omega\sin(\omega t + \phi)\right].
\end{aligned}$$

Substituting them into the equation of motion and collecting terms with cos and sin gives

$$0 = Ae^{\lambda t}\left\{\left[m\left(\lambda^2 - \omega^2\right) - \lambda\gamma + D\right]\cos(\omega t + \phi) + \left[2m\lambda\omega - \gamma\omega\right]\sin(\omega t + \phi)\right\}.$$

The square bracket in front of sin(...) vanishes if

$$\lambda = \frac{\gamma}{2m}.$$

Then the square bracket in front of cos(...) vanishes if

$$\lambda^2 - \omega^2 = \frac{\lambda\gamma - D}{m} \quad \Leftrightarrow \quad \omega = \pm\sqrt{\lambda^2 - \frac{\lambda\gamma - D}{m}} = \pm\sqrt{\frac{D}{m} - \frac{\gamma^2}{4m^2}}.$$

3. Physical Pendulum

At time $t = 0$ a physical pendulum with moment of inertia $\Theta = 0.22\,\mathrm{kg\,m^2}$ and mass $m = 500\,\mathrm{g}$ makes an angle $\alpha_0 = 7°$ (more precisely: the line connecting rotation axis and center of mass makes this angle) with the vertical and has an angular velocity $\omega_0 > 5.75°/\mathrm{s}$ (it is moving counterclockwise). Its rotation axis and center of mass have a distance $l = 10\,\mathrm{cm}$.

(a) State the equation of motion (for $\alpha(t)$) and approximate it for small angles α.

(b) Solve the equation of motion by using the general ansatz and the *two* initial conditions.

(c) What is the amplitude of the oscillation (in degrees)? What is the period of oscillation?

(d) At which time t_1 will the pendulum reach maximal deflection (maximal α) for the first time? At which time t_2 will it do so for the second time (this time deflection will be negative)?

(e) Assuming that the pendulum has been oscillating for quite some while (before $t = 0$), at which time t_0 was the angle $\alpha = 0$ the last time before $t = 0$?

(f) Sketch the solution.

Solution:

(a) With our sign convention moment of torque and angular momentum read

$$M(t) = -mgl\sin\alpha(t), \quad L(t) = \Theta\omega(t) = \Theta\dot{\alpha}(t).$$

Using the angular version of Newton's second law, $L = \dot{M}$, we obtain for the equation of motion:

$$\Theta\ddot{\alpha}(t) + mgl\sin(\alpha(t)) = 0.$$

Assuming that the angle α is small (for all times t), we may approximate the equation:

$$\ddot{\alpha}(t) + \frac{mgl}{\Theta}\alpha(t) = 0.$$

(b) Expecting harmonic oscillation, we use the ansatz

$$\alpha(t) = A\cos(\omega t + \phi).$$

Then,

$$\omega = \sqrt{\frac{mgl}{\Theta}} = 1.5/\text{s} \quad \text{and} \quad \dot{\alpha}(t) = -A\omega\sin(\omega t + \phi).$$

The two unknown quantities A and ϕ are determined by the two initial conditions:

$$\alpha(0) = A\cos(\phi) = \alpha_0, \tag{1}$$
$$\dot{\alpha}(0) = -A\omega\sin(\phi) = \omega_0. \tag{2}$$

$$(2)/(1) \Rightarrow -\omega\tan(\phi) = \frac{\omega_0}{\alpha_0} \Rightarrow \phi = -\arctan\left[\frac{\omega_0}{\alpha_0}\frac{1}{\omega}\right] = -0.5,$$
$$(1) \Rightarrow A = \frac{\alpha_0}{\cos(\phi)} = 8^\circ,$$
$$\Rightarrow \boxed{\alpha(t) = 8^\circ\cos(1.5t/\text{s} - 0.5)}.$$

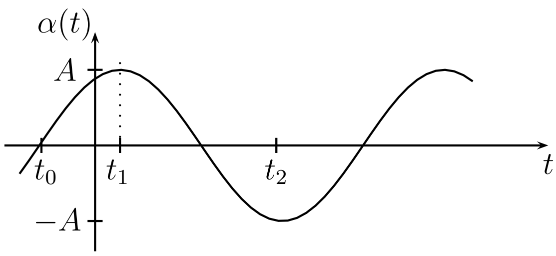
(c) $A = 8^\circ$, $T = \frac{2\pi}{\omega} = 4.2\dot{\text{s}}$.

(d) $\boxed{t_1}$ $\omega t_1 + \phi = 0 \Rightarrow t_1 = -\phi/\omega = 0.33$ s.

$\boxed{t_2}$ $t_2 = t_1 + T/2 = 2.4\,\text{s}$.

(e) $\boxed{t_0}$ $\omega t_0 + \phi = -\frac{\pi}{2} \Rightarrow t_0 = (-\frac{\pi}{2} - \phi)/\omega = -0.71\,\text{s}$.

(f)



4. Where's the Angle?

A point mass m is performing a circular motion in the two dimensional $x - y$-plane around the origin O. It moves counterclockwise with constant angular velocity ω along a circle with radius A.

(a) What is the angle $\alpha(t)$ at time t between the line connecting O with the point mass and the x-axis if for $t = 0$ this angle is ϕ?

(b) What are the coordinates $x(t)$ and $y(t)$ of the point mass at time t?

(c) Do you remember how the point mass's speed v is related to the given quantities?

(d) Just to be sure, check that $v^2 = (\dot{x}(t))^2 + (\dot{y}(t))^2$.

(e) Give the force F_{cp} which acts on the point mass. What is the component $F_{\text{cp},x}$ of this force in x-direction? Express it in terms of x (get rid of α or t!).

(f) Although it may seem pointless, since we already know the solution: Using Newton's second law $m\ddot{x} = F_{\text{cp},x}$ in x-direction, state the equation of motion for $x(t)$.

Solution:

(a) $\alpha(t) = \omega t + \phi$

(b) $x(t) = A\cos(\alpha(t)) = A\cos(\omega t + \phi), \quad y(t) = A\sin(\alpha(t)) = A\sin(\omega t + \phi)$.

(c) $v = \omega A$

(d) It is $\dot{x}(t) = -A\omega\sin(\omega t + \phi)$ and $\dot{y}(t) = A\omega\cos(\omega t + \phi)$. Since $\sin(x)^2 + \cos(x)^2 = 1$ it is

$$(\dot{x}(t))^2 + (\dot{y}(t))^2 = A^2\omega^2\sin(\omega t + \phi)^2 + A^2\omega^2\cos(\omega t + \phi)^2 = A^2\omega^2 = v^2.$$

(e) The centripetal force is $F_{\text{cp}} = mv^2/A = mA\omega^2$. Its x-component is

$$F_{\text{cp},x} = -\cos(\alpha)\ F_{\text{cp},x} = -A\cos(\alpha)\ m\omega^2 = -m\omega^2.$$

The sign ensures that if x is positive, the force points to the left, hence is negative.

(f) The equation of motion thus is $m\ddot{x} + \omega^2 x = 0$.

In the considered situation we have a circular motion and the notions of an angle $\alpha = \omega t + \phi$ and of angular velocity completely make sense. However, we have seen that if we consider only the x-coordinate, that means, if we follow the "shadow" of the point mass on the x-axis, we obtain a harmonic oscillation.

It is this analogy which justifies the notion of *angular velocity* ω for all harmonic oscillations even if there is no actual angle α involved. Usually, one then refers to α as the *phase*. As the angle $\alpha(t)$ marked the position of our point mass at a given time t, the more abstract phase $\alpha(t)$ characterizes the state of any harmonic oscillator at time t.